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Mathematical Methods for Economics - 1

BAECCC301

CENTRE FOR DISTANCE AND ONLINE EDUCATION

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MATHEMATICAL METHODS FOR ECONOMICS - 1
(BAECCC301)

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UNIT 1: THEORY OF SETS

UNIT STRUCTURE

1.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- describe sets and their representations
- identify empty set, finite and infinite sets
- define subsets, super sets, power sets, universal set
- illustrate the set operations of union, intersection, difference and complement.

1.2 INTRODUCTION

Concepts of sets, though primarily discussed in the field of Mathematics also has wider use in Economics. It was developed by the German mathematician Georg Cantor (1845-1918) developed the theory of sets and subsequently many branches of modern Mathematics have been developed based on this theory. In this unit, preliminary concepts of sets, set operations and some ideas on its practical utility will be introduced. In fact, like the number systems and functions, the set theory is very fundamental in Mathematics and hence, we need to derive its basic concepts for the study of qunatitative analysis of economics as well.

1.3 SETS AND THEIR REPRESENTATION

A set is a collection of well-defined objects. By well-defined, it is meant that given a particular collection of objects as a set and a particular object, it must be possible to determine whether that particular object is a member of the set or not.

The objects forming a set may be of any sort– they may or may not have any common property. Let us consider the following collections :

i) the collection of the prime numbers less than 15 i.e., 2, 3, 5, 7, 11, 13

ii) the collection of 0, a, Sachin Tendulkar, the river Brahmaputra

iii) the collection of the beautiful cities of India

iv) the collection of great mathematicians.

Clearly the objects in the collections (i) and (ii) are well-defined. For example, 7 is a member of (i), but 20 is not a member of (i). Similarly, ‘a’ is a member of (ii), but M. S. Dhoni is not a member. So, the collections (i) and (ii) are sets. But the collections (iii) and (iv) are not sets, since the objects in these collections are not well-defined.

The objects forming a set are called elements or members of the set. Sets are usually denoted by capital letters A, B, C, ...; X, Y, Z, ..., etc., and the elements are denoted by small letters a, b, c, ...; x, y, z, ..., etc.

If ‘a’ is an element of a set A, then we write $a \in A$ which is read as ‘a belongs to the set A’ or in short, ‘a belongs to A’. If ‘a’ is not an element of A, we write

$a \notin A$ and we read as 'a does not belong to A'.

Example 1.1: For example, let A be the set of prime number less than 15. Then $2 \in A$, $3 \in A$, $5 \in A$, $7 \in A$, $11 \in A$, $14 \in A$

$1 \notin A$, $4 \notin A$, $17 \notin A$, etc.

Representation of Sets : Sets are represented in the following two methods :

1. Roster or tabular method
2. Set-builder or Rule method

In the Roster method, the elements of a set are listed in any order, separated by commas and are enclosed within braces, For example,

A = {2, 3, 5, 7, 11, 13}

B = {0, a Sachin Tendulcar, the river Brahmaputra}

C = {1, 3, 5, 7, ...}

In the set C, the elements are all the odd natural numbers. We cannot list all the elements and hence the dots have been used showing that the list continues indefinitely.

In the Rule method, a variable x is used to represent the elements of a set, where the elements satisfy a definite property, say P(x). **Symbolically, the set is denoted by {x : P(x)} or {x | p(x)}.**

Example 1.2: A = {x : x is an odd natural number}

$B = \{x : x^2 - 3x + 2 = 0\}$, etc.

If we write these two sets in the Roster method, we get,

A = {1, 3, 5, ...}

B = {1, 2}

Some Standard Symbols for Sets and Numbers : The following standard symbols are used to represent different sets of numbers :

N = {1, 2, 3, 4, 5, ...}, the set of natural numbers

Z = {..., –3, –2, –1, 0, 1, 2, 3, ...}, the set of integers

Q = $\{x : x = {}^p/_q;\ p, q \in Z, q \neq 0\}$, the set of rational numbers

R = {x : x is a real number}, the set of real numbers

Z^+, Q^+, R^+ respectively represent the sets of positive integers, positive rational numbers and positive real numbers. Similarly Z^-, Q^-, R^- represent respectively the sets of negative integers, negative rational numbers and

1) It should be noted that the symbol ':' or '|' stands for the phrase 'such that'.
2) While writing a set in Roster method, only distinct elements are listed. For example, if A is the set of the letters of the word MATHEMATICS, then we write
A = {A, E, C, M, H, T, S, I}
The elements may be listed in any order.

negative real numbers. Z^0, Q^0, R^0 represent the sets of non-zero integers, non-zero rational numbers and non-zero real numbers.

Illustrative Examples :

1.3: Examine which of the following collections are sets and which are not :

i) the vowels of the English alphabet
ii) the divisors of 56
iii) the brilliant students degree-course of Guwahati
iv) the renowned cricketers of Assam.

Solution :

i) It is a set, V = {a, e, i, o, u}
ii) It is a set, D = {1, 2, 4, 7, 8, 14, 28, 56}
iii) not a set, elements are not well-defined.
iv) not a set, elements are not well-defined.

1.4: Write the following sets in Roster method :

i) the set of even natural numbers less than 10
ii) the set of the roots of the equation $x^2–5x+6 = 0$
iii) the set of the letters of the word EXAMINATION

Solution :

i) {2, 4, 6, 8}
ii) {2, 3}
iii) {E, X, A, M, I, N, T, O}

1.5: Write the following sets in Rule method :

i) E = {2, 4, 6, ...}
ii) A = {2, 4, 8, 16, 32}
iii) B = {1, 8, 27, 64, 125, 216}

Solution :

i) $E = \{x : x = 2n, n \in N\}$
ii) $A = \{x : x = 2^n, n \in N, n < 6\}$
iii) $B = \{x : x = n^3, n \in N, n \leq 6\}$

CHECK YOUR PROGRESS

Q 1: Express the following sets in Roster method :

i) A = {x : x is a day of the week}

ii) B = {x : x is a month of the year}

iii) C = {x : $x^3 - 1 = 0$}

iv) D = {x : x is a positive divisor of 100}

v) E = {x : x is a letter of the word ALGEBRA}

Q 2: Express the following sets in Set-builder method :

i) A = {January, March, May, July, August, October, December}

ii) B = {0, 3, 8, 15, 24, ...}

iii) C = {0, ±5, ±10, ±15, ...}

iv) D = {a, b, c, ..., x, y, z}

Q 3: Write true or false :

i) $5 \in N$ ii) $½ \in Z$ iii) $-1 \in Q$

iv) $\sqrt{2} \in R$ v) $\sqrt{-1} \in R$ vi) $-3 \notin N$

1.4 THE EMPTY SET

Definition : A set which does not contain any element is called an empty set or a null set or a void set. It is denoted by ϕ.

The following sets are some examples of empty sets.

i) the set {x : $x^2 = 3$ and $x \in Q$}

ii) the set of people in Assam who are older than 500 years

iii) the set of real roots of the equation $x^2 + 4 = 0$

iv) the set of Lady President of India born in Assam.

1.5 FINITE AND INFINITE SETS

Let us consider the sets

A = {1, 2, 3, 4, 5}

and B = {1, 4, 7, 10, 13, ...}

If we count the members (all distinct) of these sets, then the counting process comes to an end for the elements of set A, whereas for the elements of B, the counting process does not come to an end. In the first case we say that A is a finite set and in the second case, B is called an infinite set. A has finite number of elements and number of elements in B are infinite.

A ***finite set*** can always be expressed in roster method. But an ***infinite set*** cannot be always expressed in roster method as the elements may not follow a definite pattern. For example, the set of real numbers, R cannot be expressed in roster method.

Definition : A set containing finite number of distinct elements so that the process of counting the elements comes to an end after a definite stage is called a ***finite set***; otherwise, a set is called an ***infinite set***.

Example 1.6: State which of the following sets are finite and which are infinite.

i) the set of natural numbers N
ii) the set of male persons of Assam as on January 1, 2009.
iii) the set of prime numbers less than 20
iv) the set of concentric circles in a plane
v) the set of rivers on the earth.

Solution :

i) N = {1, 2, 3, ...} is an infinite set
ii) it is a finite set
iii) {2, 3, 5, 7, 11, 13, 17, 19} is a finite set
iv) it is an infinite set
v) it is a finite set.

1.6 EQUAL SETS

Definition : Two sets A and B are said to be equal sets if every element of A is an element of B and every element of B is also an element of A. In otherwords, A is equal to B, denoted by A = B if A and B have exactly the same elements. If A and B are not equal, we write $A \neq B$.

Let us consider the sets

A = {1, 2}

B = {x : (x–1)(x–2) = 0}

C = {x : (x–1)(x–2)(x–3) = 0}

Clearly B = {1, 2}, C = {1, 2, 3} and hence A = B, $A \neq C$, $B \neq C$.

Example 2.7 : Find the equal and unequal sets :

i) A = {1, 4, 9}
ii) B = $\{1^2, 2^2, 3^3\}$
iii) C = {x : x is a letter of the word TEAM}

iv) D = {x : x is a letter of the word MEAT}

v) E = {1, {4}, 9}

Solution : $A = B$, $C = D$, $A \neq C$, $A \neq D$, $A \neq E$, $B \neq C$, $B \neq D$, $B \neq E$, $C \neq E$, $D \neq E$

1.7 SUBSETS, SUPERSETS, PROPER SUBSETS

Let us consider the sets A = {1, 2, 3}, B = {1, 2, 3, 4} and C = {3, 2, 1}. Clearly, every element of A is an element of B, but A is not equal to B. Again, every element of A is an element of C, and also A is equal to C. In both cases, we say that A is a subset of B and C. In particular, we say that A is a proper subset of B, but A is not a proper subset of C.

Definition : If every element of a set A is also an element of another set B, then A is called a subset of B, or A is said to be contained in B, and is denoted by $A \subseteq B$. Equivalently, we say that B contains A or B is a **superset** of A and is denoted by $B \supseteq A$. Symbolically, $A \subseteq B$ means that
for all x, if $x \in A$ then $x \in B$.

If A is a subset of B, but there exists atleast one element in B which is not in A, then A is called a proper subset of B, denoted by $A \subset B$. In otherwords, $A \subset B \Leftrightarrow (A \subseteq B \text{ and } A \neq B)$.

The symbol '$\Leftrightarrow$' stands for 'logically implies and is implied by' (see unit 10).

Some examples of proper subsets are as follows :

$N \subset Z$, $N \subset Q$, $N \subset R$,

$Z \subset Q$, $Z \subset R$, $Q \subset R$.

It should be noted that any set A is a subset of itself, that is, $A \subseteq A$. Also, the null set ϕ is a subset of every set, that is, $\phi \subseteq A$ for any set A. Because, if $\phi \subseteq A$, then there must exist an element $x \in \phi$ such that $x \notin A$. But $x \notin \phi$, hence we must accept that $\phi \subseteq A$.

Combining the definitions of equality of sets and that of subsets, we get $A = B \Leftrightarrow (A \subseteq B \text{ and } B \subseteq A)$

Illustrative Examples :

1.8: Write true or false :

i) $1 \subset \{1, 2, 3\}$

According to equality of sets discussed above, the sets
A = {1, 2, 3} and
B = {1, 2, 2, 2, 3, 1, 3}
are equal, since every member of A is a member of B and also every member of B is a member of A. This is why identical elements are taken once only while writing a set in the Roster method.

ii) $\{1, 2\} \subseteq \{1, 2, 3\}$

iii) $\phi \subseteq \{\{\phi\}\}$

iv) $\phi \subseteq \{\phi, \{1\}, \{a\}\}$

v) $\{a, \{b\}, c, d\} \subset \{a, b, \{c\}, d\}$

Solution :

i) False, since $1 \in \{1, 2, 3\}$.

ii) True, since every element of {1, 2} is an element of {1, 2, 3}.

iii) False, since ϕ is not an element of $\{\{\phi\}\}$.

iv) True, since ϕ is subset of every set.

v) False, since $\{b\} \notin \{a, b, \{c\}, d\}$ and $c \notin \{a, b, \{c\}, d\}$.

1.8 POWER SET

Let us consider a set A = {a, b}. A question automatically comes to our mind– 'What are the subsets of A?' The subsets of A are ϕ, {a}, {b} and A itself.

These subsets, taken as elements, again form a set. Such a set is called the power set of the given set A.

Definition : The set consisting of all the subsets of a given set A as its elements, is called the power set of A and is denoted by P(A) or 2^A.

Thus, P(A) or $2^A = \{X : X \subseteq A\}$

Clearly,

i) $P(\phi) = \{\phi\}$

ii) if A = {1}, then $P^A = \{\phi, \{1\}\}$

iii) if A = {1, 2}, then $P^A = \{\phi, \{1\}, \{2\}, A\}$

iv) if A = {1, 2, 3}, then $P^A = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, A\}$

From these examples we can conclude that if a set A has n elements, then P(A) has 2^n elements.

1.9 UNIVERSAL SET

A set is called a Universal Set or the Universal discourse if it contains all the sets under consideration in a particular discussion. A universal set is denoted by U.

Example 1.9:

i) For the sets N={1, 2, 3}, Z={3, 7, 8} and Q = {4, 5, 6, 9}

We can take U = {1, 2, 3, 4, 5, 6, 7, 8, 9}

ii) In connection with the sets N, Z, Q we can take U as the universal set.

iii) In connection with the population census in India, the set of all people in India is the universal set, etc.

CHECK YOUR PROGRESS

Q 4: Find the empty sets, finite and infinite sets:

i) the set of numbers divisible by zero

ii) the set of positive integers less than 15 and divisible by 17

iii) the set of planets of the solar system

iv) the set of positive integers divisible by 4

v) the set of Olympians from Assam participating in 2008, Beijing Olympic.

Q 5: Examine the equality of the following sets :

i) $A = \{2, 3\}, B = \{x : x^2 - 5x + 6 = 0\}$

ii) A = {x : x is a letter of the word WOLF}

B = {x : x is a letter of the word FLOW}

iii) $A = \{a, b, c\}, B = \{a, \{b, c\}\}$

Q 6: Write true or false :

i) $\{1, 3, 5\} \subseteq \{5, 1, 3\}$ ii) $\{a\} \subset \{\{a\}, b\}$

iii) $\{x : (x-1)(x-2) = 0\} \not\subset \{x : (x^2-3x+2)(x-3) = 0\}$

Q 7: Write down the power sets of the following sets :

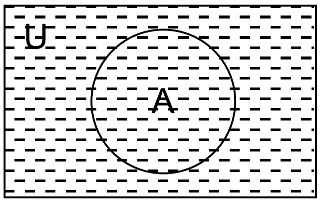
i) $A = \{1, 2, 3, 4\}$ ii) $B = \{1, \{2, 3\}\}$

Q 8: Give examples to show that $(A \subseteq B \text{ and } B \subseteq C) \Rightarrow A \subseteq C$.

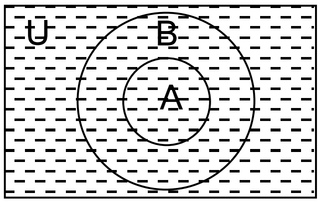
1.10 VENN DIAGRAM

Simple plane geometrical areas are used to represent relationships between sets in meaningful and illustrative ways. These diagrams are called Venn-Euler diagrams, or simply the Venn-diagrams.

In Venn diagrams, the universal set U is generally represented by a set of points in a rectangular area and the subsets are represented by circular regions within the rectangle, or by any closed curve within the rectangle. As an illustration Venn diagrams of $A \subset U$, $A \subset B \subset U$ are given below :



$A \subset U$



$A \subset B \subset U$

Similar Venn diagrams will be used in subsequent discussions illustrating different algebraic operations on sets.

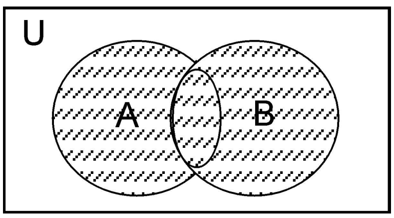
1.11 SET OPERATIONS

We know that given a pair of numbers x and y, we can get new numbers $x + y$, $x - y$, xy, $^{x}/_{y}$ (with $y \neq 0$) under the operations of addition, subtraction, multiplication and division. Similarly, given the two sets A and B we can form new sets under set operations of union, intersection, difference and complements. We will now define these set operations, and the new sets thus obtained will be shown with the help of Venn diagrams.

1.11.1 Union of Sets

Definition : The union of two sets A and B is the set of all elements which are members of set A or set B or both. It is denoted by $A \cup B$, read as 'A union B' where '$\cup$' is the symbol for the operation of 'union'. Symbolically we can describe $A \cup B$ as follows :

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$



$A \cup B$ (Shaded)

It is obvious that $A \subseteq A \cup B$, $B \subseteq A \cup B$

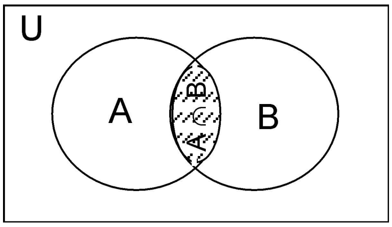
Example 1.10 : Let A = {1, 2, 3, 4}, B = {2, 4, 5, 6}

Then $A \cup B$ = {1, 2, 3, 4, 5, 6}

1.11.2 Intersection of Sets

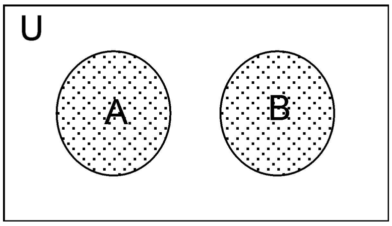
Definition : The intersection of two sets A and B is the set of all elements which are members of both A and B. It is denoted by $A \cap B$, read as 'A intersections B', where '$\cap$' is the symbol for the operation of 'intersection'. Symbolically we can describe it as follows:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$



$A \cap B$ (Shaded)

From definition it is clear that if A and B have no common element, then $A \cap B = \phi$. In this case, the two sets A and B are called disjoint sets.



$A \cap B = \phi$

It is obvious that $A \cap B \subseteq A$, $A \cap B \subseteq B$.

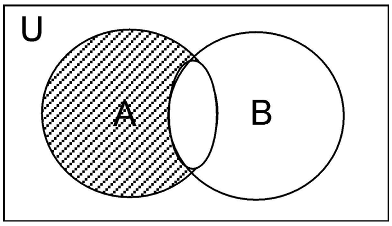
Example 1.11 : Let A = {a, b, c, d}, B = {b, d, 4, 5}
Then $A \cap B = \{b, d\}$

Example 1.12 : Let A = {1, 2, 3}, B = {4, 5, 6}
Then $A \cap B = \phi$.

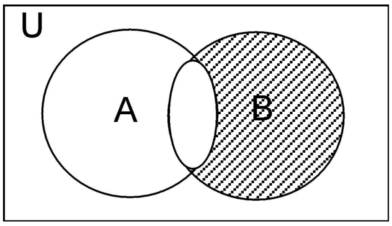
1.11.3 Difference of Sets

Definition : The difference of two sets A and B is the set of all elements which are members of A, but not of B. It is denoted by A–B. Symbollically, $A - B = \{x : x \in A \text{ and } x \notin B\}$

Similarly, $B - A = \{x : x \in B \text{ and } x \notin A\}$



A – B (Shaded)



B – A (Shaded)

Example 1.13: Let A = {1, 2, 3, 4, 5}, B = {1, 4, 5}, C = {6, 7, 8}

Then $A - B = \{2, 3\}$

$A - C = A$

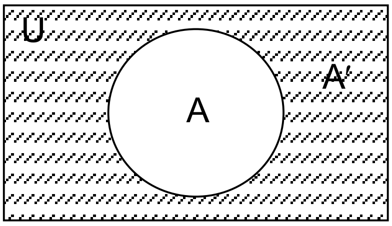
$B - C = B$

$B - A = \phi$

1.11.4 Complement of a Set

Definition : If U be the universal set of a set A, then the set of all those elements in U which are not members of A is called the **Compliment** of A, denoted by A^C or A'.

Symbolically, $A' = \{x : x \in U \text{ and } x \notin A\}$.



A′ (Shaded)

Clearly, $A' = U - A$.

Example 1.14: Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9} and A = {2, 4, 6, 8}

Then $A' = \{1, 3, 5, 7, 9\}$ **Illustrative**

Examples :

Example 1.15: If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{2, 4, 6, 8, 10\}$

$B = \{3, 6, 9\}$

and $C = \{1, 2, 3, 4, 5\}$, then find

(i) $A \cup B$, (ii) $A \cap C$, (iii) $B \cap C$, (iv) A', (v) $A \cup B'$, (vi) $C' \cap B$, (vii) $A' \cup C'$, (viii) $A - C$, (ix) $A - (B \cup C)'$, (x) $A' \cap B'$.

Solution : i) $A \cup B = (2, 3, 4, 6, 8, 9, 10\}$

ii) $A \cap C = \{2, 4\}$

iii) $B \cap C = \{3\}$

iv) $A' = \{1, 3, 5, 7, 9\}$

v) $B' = \{1, 2, 4, 5, 7, 8, 10\}$

So, $A \cup B' = \{1, 2, 4, 5, 6, 7, 8, 10\}$

vi) $C' = \{6, 7, 8, 9, 10\}$

So, $C' \cap B = \{6, 9\}$

vii) From (iv) & (vi), $A' \cup C' = \{1, 3, 5, 6, 7, 8, 9, 10\}$

viii) $A - C = \{6, 8, 10\}$

ix) $B \cup C = \{1, 2, 3, 4, 5, 6, 9\}$

$(B \cup C)' = \{7, 8, 10\}$

So, $A - (B \cup C)' = \{2, 4, 6\}$

x) From (iv) & (v), $A' \cap B' = \{1, 5, 7\}$.

CHECK YOUR PROGRESS

Q 9: If $A = \{a, b, c\}$, $B = \{c, d, e\}$, $U = \{a, b, c, d, e, f\}$ then find

i) $A \cup B$ ii) $A \cap B$ iii) $A - B$ iv) $B - A$ v) A'

1.12 LET US SUM UP

- A set is a collection of well-defined and distinct objects. The objects are called members or elements of the set.
- Sets are represented by capital letters and elements by small letters. If 'a' is an element of set A, we write $a \in A$, otherwise $a \notin A$.
- Sets are represented by (i) Roster or Tabular method and (ii) Rule or Set-builder method.
- A set having no element is called empty set or null set or void set, denoted by ϕ.
- A set having a finite number of elements is called a finite set, otherwise it is called an infinite set.

- Two sets A and B are equal, i.e. A = B if and only if every element of A is an element of B and also every element of B is an element of A, otherwise $A \neq B$.
- A is a subset of B, denoted by $A \subseteq B$ if every element of A is an element of B and A is a proper subset of B if $A \subseteq B$ and $A \neq B$. In this case, we write $A \subset B$.
- A = B if and only if $A \subseteq B$ and $B \subseteq A$.
- The set of all the subsets 8 a set A is called the power set of A, denoted by P(A) or 2^A. If $|A| = n$, then $|P(A)| = 2^n$.
- Venn diagrams are plane geometrical diagrams used for representing relationships between sets.
- The union of two sets A and B is $A \cup B$ which consists of all elements which are either in A or B or in both. $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- The intersection of two sets A and B is $A \cap B$ which consists of all the elements common to both A and B.
- For any two sets A and B, the difference set, A – B consists of all elements which are in A, but not in B. $A - B = \{x : x \in A \text{ and } x \notin B\}$
- The Universal set U is that set which contains all the sets under any particular discussion as its subsets.
- The complement of a set A, denoted by A^c or A' is that set which consists of all those elements in U which are not in A.

  $A' = \{x : x \in U \text{ and } x \notin A\} = U - A$

1.14 FURTHER READING

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1.14 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: i)A = {Monday, Tuesday, Wednessday, Thursday, Friday, Saturday, Sunday}

ii) B = {January, February, March, April, May, June, July, August, September, October, November, December}

iii) C = {1, w, w^2}

iv) D = {1, 2, 4, 5, 10, 20, 25, 50, 100}

v) E = {A, B, E, G, L, R}

Ans to Q No 2: i)A = {x : x is a month of the year having 31 days}

ii) $B = \{x : x = n^2 - 1, n \in N\}$

iii) $C = \{x : x = 5n, n \in Z\}$

iv) D = {x : x is a letter of the English Alphabet}

Ans to Q No 3: i) True, ii) False, iii) True, iv) True, v) False, vi) True.

Ans to Q No 4: i) ϕ, ii) ϕ, iii) finite, iv) infinite, v) ϕ.

Ans to Q No 5: i) B = {2, 3} = A

ii) A = {W, O, L, F}, B = {F, L, O, W} and so, A = B

iii) $A \neq B$; since $b \in A$ but $b \notin B$.

Ans to Q No 6: i)True

ii) False, since $\{a\} \in \{\{a\}, b\}$

iii) $\{x : (x-1)(x-2) = 0\} = \{1, 2\}$, $\{x : (x^2-3x+2)(x-3) = 0\} = \{1, 2, 3\}$

Hence $\{x : (x-1)(x-2) = 0 \subset \{x : (x^2-3x+2)(x-3) = 0\}$ and so, the given result is false.

Ans to Q No 7: i) P(A) = $\{\phi$, {1}, {2}, {3}, {4}, {1, 2}, {1, 3}, {1, 4},
{2, 3}, {2, 4}, {3, 4},
{1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}, A}

ii) P(B) = {ϕ, {1}, {{2, 3}}, B}

Ans to Q No 8: Let A = {1, 2}, B = {0, 1, 2, 3}, C = {0, 1, 2, 3, 4, 5, 7}

Ans to Q No 9: i) $A \cap B$ = {a, b, c, d, e}; ii) {C}; iii) A – B = {a, b},
iv) B – A = {d, e}; v) A′ = {d, e, f}

1.15 MODEL QUESTIONS

Q 1: Give examples of i) five null sets

ii) five finite sets

iii) five infinite sets

Q 2: Write down the following sets in rule method :

i) $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$

ii) $B = \left\{\frac{1}{1.2}, \frac{1}{1.3}, \frac{1}{3.4}, \frac{1}{4.5}, \ldots\right\}$

iii) C = {2, 5, 10, 17, 26, 37, 50}

Q 3: Write down the following sets in roster method

i) A = {x : x $\in$ N, 2 < x < 10}

ii) B = {x : x $\in$ N, 4+x < 15}

iii) C = {x : x $\in$ Z, $-5 \le x \le 5$}

Q 4: If A = {1, 3}, B = {1, 3, 5, 9}, C = {2, 4, 6, 8} and
D = {1, 3, 5, 7, 9} then fill up the dots by the symbol $\subseteq$ or $\not\subset$:

i) A ... B, ii) A ... C, iii) C ... D, iv) B ... D

Q 5: Write true or false :

i) 4 $\in$ {1, 2, {3, 4}, 5}, (ii) ϕ = {ϕ}

iii) A = {2, 3} is a proper subset of B = {x : (x–1)(x–2)(x–3) = 0}

iv) $A \subseteq B$, $B \subseteq C \Rightarrow A \subseteq C$

Q 6: If U = {x : x $\in$ N}, A = {x : x $\in$ N, x is even}, B = {x : x $\in$ N, x < 10}
C = {x : x $\in$ N, x is divisible by 3}, then find

i) $A \cup B$, ii) $A \cap C$, iii) $B \cap C$, iv) A′, (v) B′, vi) C′.

Q 7: If $A \cup B$ = B and $B \cup C$ = C, then show that $A \subseteq C$.

Q 8: If U = {–5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5},
A = {–5, –2, 1, 2, 4}

$B = \{-2, -3, 0, 2, 4, 5\}$

$C = \{1, 0, 2, 3, 4, 5\}$,

then find i) $A \cup B$, ii) $A \cap C$, iii) $A \cap (B \cup C)$, iv) $B \cap C'$,

v) $A' \cup (B \cap C')$, vi) $A - C'$, vii) $A - (B \cup C)'$, viii) $(B \cup C')$,

ix) $A' \cup C'$, x) $(C' \cup B) - A$.

Q 9: Prove the following :

i) If A, B, C are three sets such that $A \subseteq B$,

then $A \cup C \subseteq B \cup C$, $A \cap C \subseteq B \cap C$.

ii) $A \subseteq B$ if and only if $B' \subseteq A'$.

iii) $A \subseteq B$ if and only if $A \cap B = A$.

iv) If $A \cap B = \phi$, then $A \subseteq B'$.

Q 10: How many elements are there in P(A) if A has

i) 5 elements, ii) 2^n elements?

*** ***** ***

UNIT 2: NUMBERS, RELATIONS AND FUNCTIONS

UNIT STRUCTURE

2.1 LEARNING OBJECTIVES

After going through this unit, you will be able to-

- identify various types of numbers
- express a rational number as a terminating or non-terminating repeating decimal
- know imaginary and complex numbers
- know about constants,variables and its types
- derive knowledge about the concept of a relation
- describe the concept of a function
- discuss the different types of function.

2.2 INTRODUCTION

Creation of numbers is the greatest inventions in the history of mankind.Numbers, which are a basic part of mathematics, help us to understand algebra, to measure geometric objects, and to make predictions using probability and statistics. In this unit,we will discuss diferent types numbers i.e Natural number,Whole number,Integers,Rational number,Irrational number,real number,imaginary number and Complex number.We will also discuss about the concept of constants and variables.

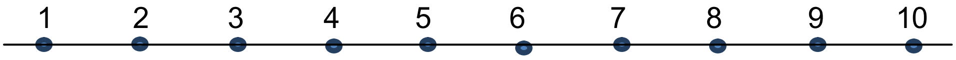
In this unit we will aslo discuss the concept of relations and functions. We shall define the concept of relations and functions. Apart from those, we shall also discuss the different types of functions in the general mathematial context. Later, in Unit 3, we shall discuss different functions in the particular context of Economics.

2.3 NATURAL NUMBER (OR COUNTING NUMBER)

Counting things is easy for us. We can count objects in large numbers, for example, the number of students in the college, and represent them through numerals. When we begin to count, we use 1, 2, 3, 4, ...

These numbers come our mind naturally when we are counting. Numbers such as 1, 2, 3, 4, 5, 6, ……. .etc. which are used in day-to-day counting of things are called ***Natural Numbers*** or ***Counting Numbers***.

Gemetrical representation of Natural numbers



2.4 WHOLE NUMBER

The natural number along with zero form the collection of whole number. When 0 (zero) is included into the system of the natural number, the resultant extended system is called the system of whole numbers. Numbers such as 0, 1, 2, 3, 4, 5, 6, ……. . etc are called whole Numbers. The set of ***Whole number*** is denoted by W.

2.5 INTEGERS

There are times when we need to use numbers with a negative sign. For example, involving opposite measurements, such as, profit and loss in business, fluctuation of temperature, altitude or depth of places water level in lake or river, level of oil in tanketc. This is when we want to go below zero on the number line. These are called negative numbers.

Therefore, the negative of the natural numbers, zero and the natural numbers together constitute the integers.

The collection of numbers …. , -4, -3, -2, -1, 0, 1, 2, 3, 4, ….. is called ***integers***. So, -1, -2, -3, -4, ……. Called negative numbers are ***negative integers*** and 1, 2, 3, 4, …. . called positive numbers are positive integers.

2.6 RATIONAL NUMBERS OR FRACTIONS

A fraction is a number representing part of a whole. The whole may be a single object or a group of objects. If x and y are two natural numbers, then $\frac{x}{y}(y \neq 0)$ is a fraction where, x and y are respectively the numerator and the denominator of the fraction.

For examples : $\frac{2}{3}, \frac{4}{9}, \frac{123}{23}$ etc.

Types of fractions :

(1) **Proper fraction :** If in a fraction, the numerator is always less than the denominator, then fraction is called *proper fraction*. For example : $\frac{3}{4}, \frac{1}{2}, \frac{9}{10}, \frac{5}{8}$ etc are all proper fractions.

(2) **Improper fraction :** The fractions, where the numerator is bigger than the denominator are called *Improper fractions*. For example : $\frac{13}{12}, \frac{12}{7}, \frac{18}{5}$ etc are all improper fractions.

(3) **Mixed fraction :** A *mixed fraction* has a combination of a whole and a part. We can express an improper fraction as a mixed fraction by dividing the numerator by denominator to obtain the quotient and the remainder. Thus, the mixed fraction can be written as Quotient $\frac{\text{Remainder}}{\text{Divisor}}$.

For example : $2\frac{3}{4}, 7\frac{1}{9}, 5\frac{3}{7}$ etc are all mixed fractions. These mixed fractions can be expressed as improper fractions. i.e., $2\frac{3}{4} = 2 + \frac{3}{4} = \frac{2 \times 4}{4} + \frac{3}{4} = \frac{11}{4}$.

2.7 IRRATIONAL NUMBERS

An irrational number is a number that cannot be expressed as afraction for any integers. A number which can be neither be expressed as a terminating decimal nor as a repeating decimal, is called an irrational number. Thus, nonterminating, nonrepeating decimals are irrational number.

Clearly, 10.01001000100001…… is a non-terminating ,non-repeating decimals, and therefore, it is irrational.

Irrational Numbers cannot be expressed as the quotient of two integers (ie a fraction) such that the denominator is not zero

Irrational numbers have decimal expansions that neither terminate nor repeating. All numbers that are not rational are considered irrational. An irrational number can be written as a decimal, but not as a fraction. An irrational number has endless non-repeating digits to the right of the decimal point.

One example of irrational number is the Pi(π). Pi(π) is a famous irrational number.

The value of Pi (π) is equal to 3. 14159265358979323846264338327950 (which is a non-terminating and non-repeating decimal).

2.8 REAL NUMBERS

A number whose square is non-negative is called a real number. In fact, all rational and all irrational numbers form the set of all real numbers. Every real number is either rational or irrational.

CHECK YOUR PROGRESS

Q 1: Convert each of the following fractions to decimals

(a) 1/4(b) 2/3 (c)4/5 (d) 3/7

Q 2: Classify the following numbers as rational or irrational, terminating or recurring decimals :

(a) $\sqrt{100}$ (b) $\frac{1}{\pi}$(c) $\frac{4}{5}$ (d) 0.75 (e) $\frac{13}{99}$

Q 3: How many rational numbers and irrational numbers lie between the numbers 2 and 7?

2.9 IMAGINARY NUMBERS

An imaginary number is defined as any number that, when squared, results in a real number less than zero. When any real number is squared, the result is never negative, however, the square of an imaginary number is always negative. Imaginary numbers are written using the variable i.

Imaginary numbers have the form biwhere b is a non-zero real number and i is the imaginary unit, defined such that i=$\sqrt{-1}$

2.10 COMPLEX NUMBER

A number of the form x+iy where x and y are real numbers and i= $\sqrt{-1}$, is called a complex number . It is denoted by Z .

Thus, Z=x+iy is a complex number.

x is called its real part and is denoted by Re(z). Thus , Re(z) = x and y is called its imaginary part and is denoted by Im (z). Thus ,Im(z) = y.

If x=0 , then z=0+iy is a purely (or wholly) imaginary number.

If y=0 , then z=x+i.0 which is wholly is a real number.

2.11 PRIME NUMBER

The number other than 1 whose only factors are 1 and the number itself.Such number are 2, 3, 5, 7, 11 etc. These numbers are prime numbers.

The numbers other than 1 whose only factors are 1 and the number itself are called Prime Numbers.

Note : (a) 1 is not a prime number.

2 is the smallest prime number.

2.12 CONCEPT OF CONSTANTS AND ITS DIFFERENT TYPES

A symbol which represents a definite number used in any operation is called a constant. A constant retains the same value throughout a set of mathematical operation. These are generally denoted by a, b, c etc.

For example, 7, 13 , 1/12, -6 , π , e, m , c, d etc are called constants. These quantities which donot change in any mathematical operations.

Types of Constants :

Constants are of two types :

(a) Absolute constant : A constant which remains the same throughout a set of mathematical operation is known as absolute constant. The value of absolute constant remains fixed in all conditions .All numerical numbers are absolute constant. For example, 3, 6 , 7/16 , $\sqrt{3}$, π etc. are absolute constants.

(b) **Arbitrary constant :** A constant which remains same in a particular operation, but changes with the change of reference, is called arbitrary constant. The value of arbitrary constant remains unchanged in a particular problem.

For example, In coordinate geometry of two dimension the equation y = mx + c represents a straight line. Here But m and c are constants, but they are different for different straight lines. Therefore, m and c are arbitrary constants.

2.13 CONCEPT OF VARIABLE AND ITS DIFFERENT TYPES

A variable is a symbol which can assume any value out of a given set values. The quantities, like height, weight, time, temperature, profit, sales etc, are examples of variables. The variables are usually denoted by x, y, z, u, v, w etc.

Types of Variable : There are two types of variables :

(a) **Independent Variable :** A variable which can take any arbitrary value, is called independent variable.

(b) **Dependent Variable :** A variable whose value depends upon the independent variable is called dependent variable.

For example:

(i) $y = x^2$ if x=2 then y=4, so value of y depends on x. Here y is dependent and x is independent variable.

(ii) Let A be the area and r be the radius of a circle. Then $A = \pi r^2$ where A depends on r i. e area of a circle depends on radius of the circle. Here A is dependent and r is independent variable.

CHECK YOUR PROGRESS

Q 4: What is the smallest prime number ?

Q 5: Find the value of i^{65}

Q 6: Define constants with examples.

Q 7: Define variables with examples.

2.14 CONCEPT OF RELATION

Let us consider the following sentences.

(1) 11 is greater then 10.

(2) 35 is divisible by 7.

(3) New Delhi is the capital of India.

In each of the sentences there is a relation between two ‘objects’.

Now let us see what is meant by relation in set theory.

The concept of Set has been discussed in Unit 2.

Definition Let A and B be two non-empty sets. A subset R of A x B is said to be a **relation** from A to B.

If A=B, then any subset of A x A is said to be a relation on A.

If R $\subseteq$ A x B, and (a, b) $\in$ R; a $\in$ A, b $\in$ B, it is also written as aRb and is read as ‘a is R related to b’.

Note 1. The set of the first components of the ordered pairs of R is called the domain and the set of the second components of the ordered pairs of R is called the range of R.

2. If A, B are finite sets and n(A)=x, n(B)=y; then n(AxB)=xy. So, the number of subsets of AxB is 2^{xy}.

Therefore, the number of relations from A to B is 2^{xy}.

Example 1.1: Let A ={1, 2, 3},

B={8, 9}

$\therefore$ A x B ={(1, 8), (1, 9), (2, 8), (2, 9), (3, 8), (3, 9)}

Let R = {(1, 8), (2, 9), (3, 9)}

Clearly R $\subseteq$ AxB

$\therefore$ R is a relation from A to B.

Here 1R8, 2R9, 3R9

Domain of R = {1, 2, 3}

Range of R = {8, 9}

Example 2.2: Let X be the set of odd integers.

Let R = {(x, y) : x, y $\in$ X and x+y is odd}

We know that the sum of two odd integers is an even integer.

$\therefore$ if x, y are odd, then x+y cannot be odd.

$\therefore$ R =$\subseteq$ ϕX x X

In this case R is called a null relation on X.

Example 2.3: Let E be the set of even integers.

Let R = {(x, y) : x, y $\in$ E and x+y is even}

We know that the sum of the even integers is an even integer. $\therefore$ if x, y are even, then x+y is always even.

$\therefore$ R = E x E $\subseteq$ E x E

In this case R is called a universal relation on E.

2.14.1 Identity Relation

Let A be a non-empty set.

I_A = {(a, a) : a$\in$A} $\subseteq$ A x AA

I_A is called the identity relation on A.

Example 2.4: Let A = {1, 2, 3, 4}

Then I_A ={(1, 1), (2, 2), (3, 3), (4, 4)} $\subseteq$ A x A.

Clearly I_A is the identity relation on A.

2.14.2 Inverse Relation

Let A, B be two non-empty sets. Let R be a relation from A to B, i.e. R $\subseteq$ AxB. The inverse relation of R is denoted by R^{-1}, and is defined by R^{-1}={(b, a) : (a, b)$\in$R} $\subseteq$ B x A

Clearly, domain of R^{-1} = range of R

range of R^{-1}= domain of R

Example 2.5: Let A = {1, 2, 3}

B = {5, 6}

R = {(1, 5), (2, 6), (3, 5)} $\subseteq$ A x B

R^{-1} = {(5, 1), (6, 2), (5, 3)} $\subseteq$ B x A.

CHECK YOUR PROGRESS

Q 8: Let A, B be two finite sets, and n(A)=4, n(B)=3. How many relations are there from A to B?

Q 9: Let A be a finite set such that n(A) =5. Write down the number of relations on A.

2.15 CONCEPT OF FUNCTION

A function is a technical term used to symbolise relationship between two or more variables. When two real varaibles, say, x and y are so related that that correspoding to every value of x, we get a definite value (or a set of definite values) of y, then y is said to be a function of x and symbolically we write:

$y = f(x)$.

A few examples of functions from our real life include:

- Income tax payable by a person is a function of his/her income. (t = f (Y). Here, t represents tax payable, and Y represents income.)
- Consumption is a function of disposable income. ($C = f(Y_d)$. Here C represents consumption, and Y_d represents disposable income.

Let A and B be two non-empty sets, and $f \subseteq A \times B$ such that

(i) $(x, y) \in f, \forall x \in A$ and any $y \in B$

(ii) $(x, y) \in f$ and $(x, y') \in f \Rightarrow y = y'$.

In this case f is said to be a function (or a mapping) from the set A to the set B. Symbolically we write it as $f : A \to B$.

Here A is called the domain and B is called the codomain of f.

Example 2.6 : Let A = {1, 2}, B = {7, 8, 9}

f = {(1, 8), 2, 7)} $\subseteq$ A x B.

Here, each element of A appears as the first component exactly in one of the ordered pairs of f.

$\therefore$ f is a function from A to B.

Example 2.7: Let A = {1, 2}, B = {7, 8, 9}

g = {(1, 7), (1, 9)} $\subseteq$ A x B

Here, two distinct ordered pairs have the same first component.

$\therefore$ f is not a function from A to B.

Example 2.8 : Let A = {1, 2, 3, 4}, B = {x, y, z, w}

Are the following relations from A to B be functions?

(i) f_1 = {(1, x), (1, w), (2, x), (2, z), (4, w)}

(ii) f_2 = {(1, y), (2, z), (3, x), (4, w)}

Solution : (i) No. Here two distinct ordered pairs (1, x), (1, w) have the same first component.

(ii) Yes.

Here, each element of A appears as the first component exactly in one of the ordered pairs of f_2.

Thus we see that.

Every function is a relation, but every relation is not a function.

We observe that if A and B are two non-empty sets and if each element of A is associated with a unique element of B, then the rule by which this association is made, is called a function from the set A to the set B. The rules are denoted by f, g etc. The sets A, B may be the same.

Let f be a function from A to B i.e.

$f : A \to B$. The unique element y of B that is associated with x of A is called the image of x under f. Symbolically we write it as f = f(x). x is called the pre-image of y. The set of all the images under f is called the range of f.

Example 2.9 : Let IN be the set of natural members, and Z be the set of integer and $f : IN \to Z$, $f(x) = (-1)^x$; $x \in IN$

Clearly, domain of f = IN

codomain of f = Z

Now $f(1) = (-1)^1 = -1$, $f(2) = (-1)^2 = 1$, $f(3) = (-1)^3 = -1$ and so on.

$\therefore$ range of f = {–1, 1}

2.16 DIFFERENT TYPES OF FUNCTION

Let A, B be two non-empty sets and $f : A \to B$ be a function.

1. If there is at least one element in B which is not the image of any element in A, then f is called an “into” function.
2. If each element in B is the image of at least one element in A, then f is

called an “onto” function (or a surjective function or a surjective).

Note : In case of an onto function, range of f=codomain of f.

1. If different elements in A have different images in B, then f is called a one-one function (or an injective function or an injection).
2. If two (or more) different elements in A have the same image in B, then f is called a many-one function.

2.16.1 Polynomial Function

A function of the form $f(x) = a_0 + a_1 x + a_2x^2 + ... + a_nx^n$ where n is a non-negative integer; and $a_0, a_1, a_2, ... a_n$ are real numbers, is called a polynamial function of degree n.

For example $f(x) = x^4 - 5x^3 + 2x^2 + 6x + 7$ is a polynomial function; whereas $f(x) = 5x + 9\sqrt{x} + 7$ is not a polynamial function.

Note : The domain of a polynomial function is IR, the codomain is also IR.

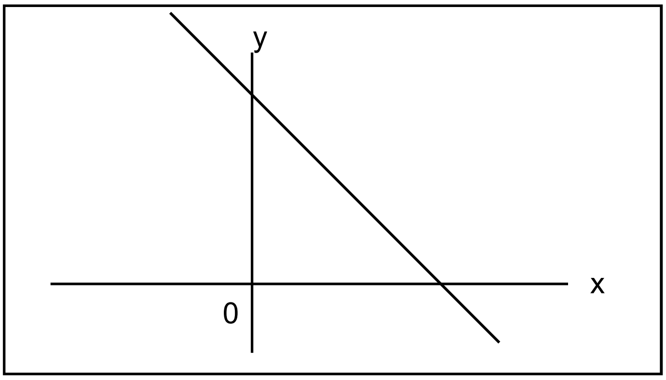
2.16.2 Linear Function

If $n = 1$,then the polynomial function is of degree 1 and is called a linear function.For $n = 1$,the function is written as $f(x) = a + bx$,where $a \& b$ are constants.

For example, f(x) = 3x + 5 is a linear function.

The graph of Linear function has been shown with the help of Figure 1.1.

Figure 2.1: Graphical Shape of a Linear Function



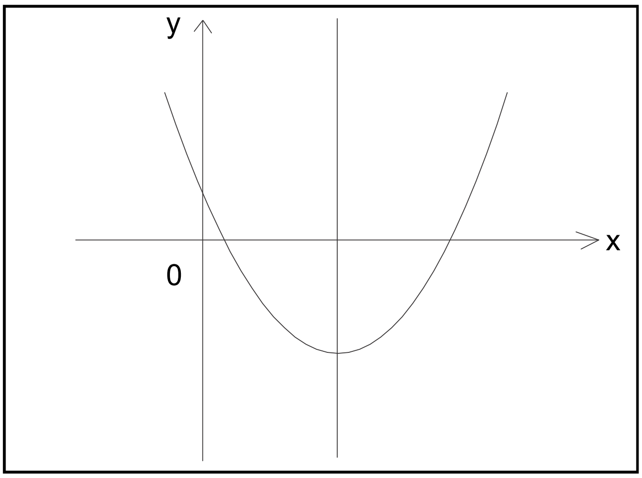
2.16.3 Qudratic Function

If $n = 2$,then the polynomial function is of degree 2 and is called a quadratic function.

A Quadratic function is of the form f(x) = $ax^2 + b\,x + c$,where a,b,c are constants.

The graph of quadratic function f(x) = $ax^2 + b\,x + c$ is parabola that opens upwards or downwards according as a>0 or a<0. It has been shown in Figure 1.2.

Figure 2.2: Graphical Shape of a Quadratic Function

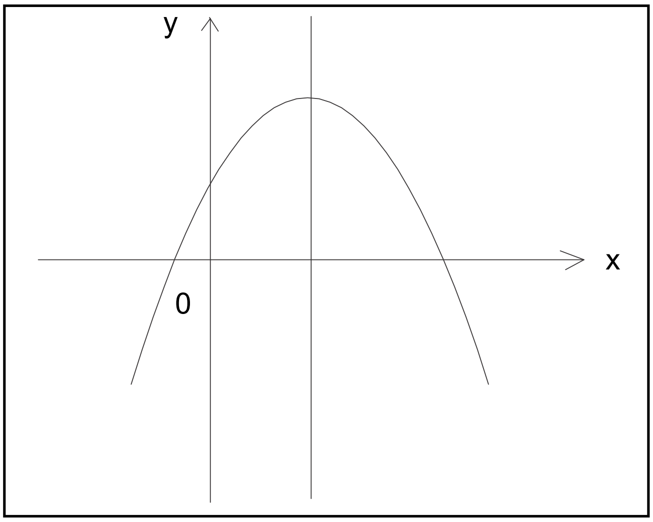


2.16.4 Cubic Function

If $n = 3$,then the polynomial function is of degree 3 and is called a cubic function.

A cubic function is of the form f(x) = $ax^3 + bx^2 + cx + d$ where a,b,c,d are constants. The graphical shape of a cubic function has been shown in Figure 2.3.

Figure 2.3: Graphical Shape of a Cubic Function



2.16.5 Power Function

A function of the form f(x) = ax^n where n is a non-negative integer; and a is constant, is called a Power function.

2.16.6 Rational Function

A function of the form y=$\frac{f(x)}{g(x)}$, where f(x) and g(x) are polynomial function and g(x) $\neq$ 0, is called a rational function.

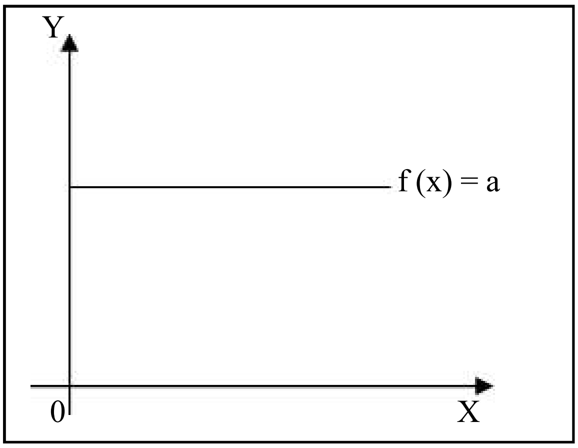
For example $\frac{x+2}{x^2-5x+6}$ a rational function. Here x $\neq$ 2, x $\neq$ 3, i.e. the domain of the rational function $\frac{x+2}{x^2-5x+6}$ is IR – {2, 3},

2.16.7 Constant Function

A function of the form f(x) = k, where k is a constant is called a constant function.

Note : The range of a constant function is a singleton set The graph of a constant function is a straight line paralled to x-axis.

Figure 2.4: Graphical Shape of a Costant Function

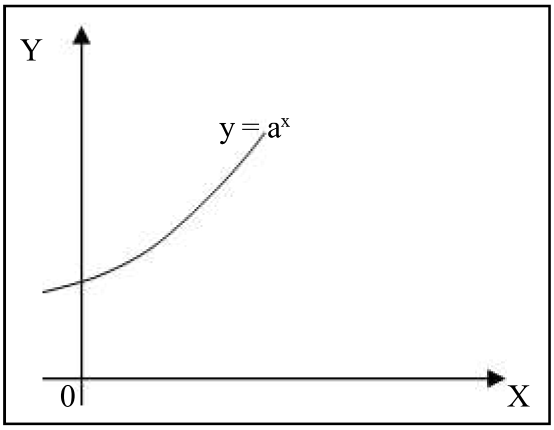


2.16.8 Exponential Function

Let a be a positive real number other than 1. then the function is of the form $y = a^x$ is called the exponential function.

Graph of an exponential function has been shown in Figure 1.5.

Figure 2.5: Graphical Shape of an Exponential Function



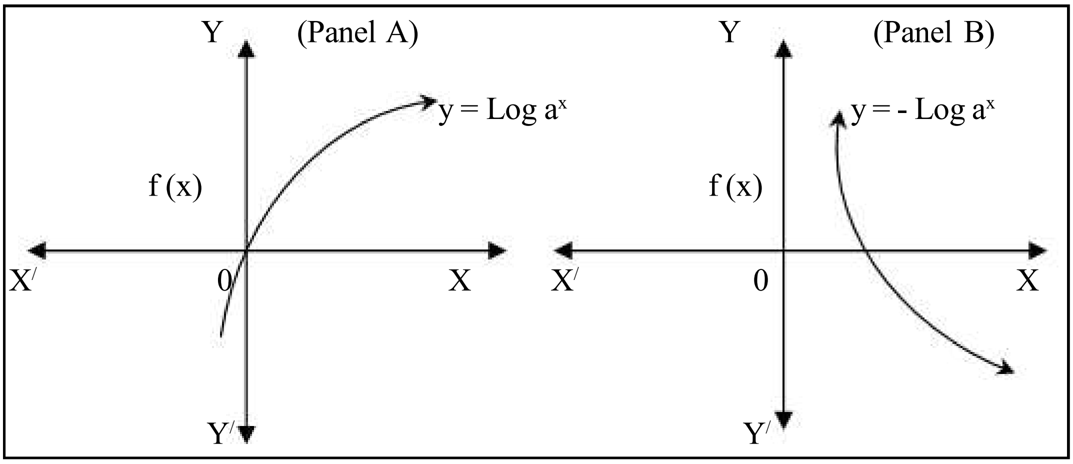
2.16.9 Logarithmic Function

Let x be a positive real number other than 1. Then the function of the form $f(x) = \log_e x$ is called the logarithmic function.Logarithmic function is inverse to exponential function.

Remark : $\log_e x$ is also denoted by lnx.

Graph of the logarithmic functions when $y = \text{Log } a^x$ and when y = -Log a^x have been shown in Panel A and Panel B of Figure 1.6 respectively.

Figure 2.6: Graphical Representation of Logarithmic Functions



2.17 LET US SUM UP

- The natural numbers along with zero form the collection of whole numbers.
- The positive natural number,zero and negative natural number together are called Integers.
- A rational number is a number which canbe put in the form p/q, where p and q are integers and $q \neq 0$.
- The decimal representation of a rational number is either terminating or non-terminatingrepeating.
- There exist infinitely many rational numbers between two rational numbers.
- The irrational numbers are decimal numbers that do not terminate and donot repeat. On calculators and in the solution of many problems,rational approximations are used to show values that are close to, but not equal to, irrational numbers.
- The sytem of rational numbers is extended to real numbers.
- Rationals and irrationals together constitute the system of real numbers.
- The numbers other than 1 whose only factors are 1 and the number itself are called Prime Numbers.

- A symbol which represents a definite number used in any operation is called a constant.
- A variable is a symbol which can assume any value out of a given set values.
- If A, B are two non-empty sets, a subset of A x B in said to be a relation from A to B.
- If A, B are two finite sets and n(A) = x, n(B) = y, the number of relations from A to B is 2^{xy}.
- If A is a non-empty set, $I_A = \{(a, a) : a \in A\}$ is called the identity relation on A.
- A relation R on a non-empty set A is called an equivalence relation if it is reflexive, symmetric and transitive.
- The inverse of an equivalence relation is also an equivalence relation.
- The intersection of two equivalence relations is also an equivalence relation.
- If A and B are two non-empty sets and if each element of A is associated with a unique element of B, then the rule by which this association is made, is called a function from A to B.
- Every function is a relation, but every relation is not a function.
- If different elements in domain have different images in codomain, then the function is one-one (injective).
- If each element in codomain is the image of at least one element in domain then the function is onto (surjective).

2.18 FURTHER READING

1) Agarwal, D.K. (2012). *Business Mathematics*, New Delhi: Vrindra Publication (p) Ltd.
2) Baruah, S. (2011). *Basic Mathematics and Its Application in Economics*, New Delhi: Trinity Press Pvt. Ltd.
3) Bose, D. (2004). *Mathematical Economics;* New Delhi: Himalaya Publishing House.

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2.19 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: (a) 1/4=0.25 (b) 2/3=0.6666.. (c) 4/5=0.8 (d) 3/7=0.4286.

Ans to Q No 2: (a) Rational (b) Irrational (c) Irrational (d) Rational (e) Irrational.

Ans to Q No 3: Infinitely many.

Ans to Q No 4: Smallest prime number is 2.

Ans to Q No 5: $i^{65} = i^{4\times16+1} = i$

Ans to Q No 6: Try it yourself.

Ans to Q No 7: Try it yourself.

2.20 MODEL QUESTIONS

Q 1: Write down an irrational number between 6 and 7.

Q 2: Prove that the following numbers are irrational

$$\sqrt{2}, \sqrt{5}, \sqrt{7}$$

Q 3: Insert 3 rational numbers between 1/5 and 3/5.

Q 4: Write 5 rational numbers and 5 irrational numbers in decimal form.

Q 5: Prove that $\sqrt{5}-\sqrt{3}$ is not a rational number.

Q 6: Which is the greater of the numbers $\frac{7}{9}$ and $\frac{8}{11}$.

Q 7: Find a rational number between 19.9 and 20.

Q 8: Let A = {1, 2, 3}, B = {3, 4, 5}.

How many relations are there from A to B ? Write down any four relations from A to B.

*** ***** ***

UNIT 3: FUNCTIONS IN ECONOMICS

UNIT STUCTURE

3.1 LEARNING OBJECTIVES

After going through this unit, you will able to

- describe a few fundamental functions in Economics.

3.2 INTRODUCTION

Functional relationship plays an important place in Economics. We often say, there is a positive relationship between price and supply of a commodity, or a negative relationship between price and demand for a commodity. This unit lays primary importance to discussing a few fundamental functions in Economics. Thus, after going through this unit, you will be able to describe a few fundamental functions in Economics.

3.3 SOME FUNDAMENTAL FUNCTIONS IN ECONOMICS

In this section, we shall discuss a few fundamental functions in Economics. Let us begin with demand Function

3.3.1 Demand Function

Demand function establishes a relationship between quantity demanded of a commodity and its determinants. The demand function for a commodity can be written as:

$D_x = f\,[\,P_x, P_s, P_c, Y, T]$

where, D_x is the quantity of X demanded, P_x is the price of the commodity X, P_s denotes the price of other commodity which can be substituted for X, P_c is the price of a commodity which is a complement of commodity X, Y is the income of the consumer, T represents other demand factors such as tastes, preferences, social custom etc. and f represents a functional relationship.

Individual demand function: It refers to the functional relationship between individual demand and the factors affectiong individual demand.

Market Demand Function: In the previous courses, we have already discussed that market demand is the summation of individual demands. Thus, the market demand function refers to the functional relationship between market demand and the factors affecting market demand. Market demand is affected by all factors affecting individual demand. In addition, it is also affected by size and composition of population, season and weather and distribution of income.

In symbolic form, a market demand function can be expressed as:

$D_m = f(P_x, P_r, Y, T, F, PD, S, D)$.

where, D_m = Market demand of commodity X; P_x = price of given commodity x; P_r = prices of related goods; Y = Income of the consumers; T = Tastes and Preferences; F = Expectation of Change in Price in future; PD = Size and Composition of population; S = Season and Weather; D = Distribution of Income.

3.3.2 Supply Function

The supply function establishes a relationship between supply of a commodity and factors that determines supply.

In symbolic form, we can express the relationship between quantity of the commodity supplied and its determinants as:

$S_x = f\,[P_x, P_s, P_c, F_j, C, T, G]$

where; S_x is quantity of the commodity supplied., P_x is its Price, Ps represents a vector of Prices of substitute goods, P_c represents a vector of prices of complementary goods. F_j represents prices of factor of production. C is the total expenditure of the consumer. T represents the state of technology. G represents goal of the producer and f represents the functional relationship.

Market Supply Function: Market supply function refers to the functional relationship between market supply and factors affecting the market supply of a commodity. Market supply is affected by all the factors affecting individual supply. In addition, it is also affected by some other factors like number of firms, future expectations regarding price and means of transportation and communication.

In symbolic form, a market supply function is expressed as:

$S_m = f\,(P_x, P_o, P_f, S_t, T, G, T, G, N, F, M)$.

where, S_m = market supply of given commodity X; P_x = price of the given commodity x; P_0 = price of other goods; P_f = prices of factors of production; S_t = state of technology; T = taxation policy; G = goals of the market; N = number of firms; F = future expectation regarding P_x; M = means of transportation and communication.

3.3.3 Utility Function

Utility function expresses how a consumer achieves maximum utility from consumption of different goods. Let us assume that a consumer consumes two commodities X and Y and so the utility function of the consumer is given by;

$U = u\,(X, Y)$

where; U = total utility, X,Y are the commodity consumed by the consumer.

When a consumer has to maximise his total utility from the consumption of a basket of goods, the Marshallian concept of consumer behaviour states that the consumer has to allocate his total budget for the consumption of various commodities in such as way that the ratio of marginal utility to price of each commodities is equal. The Hicksian concept of consumer behaviour states that in case of pair of consumer goods, the utility maximisation requires that the combination of goods should be choosen in such a way that the slope of the budget line is equal to the slope of the utility function. In other words; the budget time should be tangent to the indifference curve as well as the indifference curve must be convex to the origin.

3.3.4 Production Function

The production function is a concept that has been developed to deal with technological aspects of the theory of production. It is an embodiment of technology which yields maximum output from the given set of inputs or specifies the way in which inputs cooperate together to produce a given level of output. The general form of the production function is

$$Q = f(L, K, R, S, V, \gamma)$$

where; Q = output, L = labour input, K = capital, R = raw materials, S= land input, V = return to scale, γ= effeciency parameter.

3.3.5 Revenue Function

Revenue implies income earned by a producer from selling the output produced during a period of time. In any kind of market structure; revenue maximisation is the main objectives. However; a firm cannot pursue revenue maximisation irrespective of what happens to profit, Even with the objective of revenue maximisation, the firm must earn a certain minimum amount of profit which is sufficient enough to satisfy the share- holders. Revenue earned by a producer depends on output sold and price of that particular product.

3.3.6 Cost Function

Cost functions are derived function. They are derived from the production function which describes the available efficient methods of production at any one time. Economic theory distinguishes between short-run cost and long-run costs. Both in the short run and in the long run, total cost is a multi-variable function. Total cost is determined by many factors. Symbolically we many write the long- run cost function as:

$$C = f(x, T, P_f)$$

Where; C = Total Cost, X = Qutput, T = Technology, P_f = Price of factors.

We can also expresses the short-run cost function in mathematical form as,

$$C = f(x, T, P_f, K)$$

where K = fixed factors

It is required to be mentioned here that in case of short-run cost functions, some factors are constant which are known as fixed factors.

3.3.7 Profit Function

Profit maximization is the major objective of the producers involved in the production of a commodity. In General term, profit is nothing but the difference between revenue and cost. In symbotic form; we can express the profit function as-

$$\pi = P.Q - C.....(i)$$

Here; π = Total Product,

P = Price of the product

Q = Quantity Produced

C = Total Cost of Production.

The optimum level of output which maximises profit of a firm is arrived at when

(a) Marginal revenue equats to marginal cost and equatis

(b) Marginal cost curve cuts marginal revenue curve from below.

In order to obtain the level of output at which the profit will be maximum, we follow the procedure of maximising a function as given in equation (i) in which first derivative is zero and second derivative is negative. Now; if profit is expressed as

$\pi = R - C$

$= R(q) - C(q)$ (ii)

The first derivative of (ii) with respect to Q gives

MR = MC, here MR = Marginal revenue and MC = marginal cost.

And the second derivative of (ii) with respect to Q gives :

Slope of MR < Slope of MC

3.3.8 Consumption Function

Consumption function reveals the relationship between aggregate consumption expenditure and income. It is expressed as C= C (Y). Consumption expenditure may be related to either national income or disposable income. The consumption function is one of the cornerstones of Keynesian analysis. In most of the developed countries, consumption expenditure constitutes the single largest component of aggregate expenditure.

3.3.9 Saving Function

Saving function reveals the relationship between aggregate savings and income. It is expressed as S= S (Y). Savings may be related to either national income or disposable income. The savings function is the inverse of the consumption function. Both savings and consumption constitutes total income. Thus, Y = C + S (Here, Y = income, C = consumpiton, S = savings).

3.3.10 Investment Function

Investment decision of a firm depends on many things. If we enumerate most of the factors on which investment depends upon, the investment function may be expressed as:

$I = f(Y, r, w, Q, FMP, F, T, BC, Y_{-1}, K_{-1}, u)$

Where:

I = net investment

Y = output (or, income)

r = real interest rate

w = real wage rate

Q = ***Tobin's Q***

FMP = fiscal (tax) and monetary (credit) policies

F = financial constraints

T = technology

BC = business confidence

Y_{-1} = output of the previous year

K_{-1} = stock of capital in the previous year

u = other factors.

Please note that in the investment function stated above, most of the investments theories have been accomodated. However, it must be kept in mind that the various theories are not independent or exclusive as they have overlapping elements. For example, the Tobin's Q theory has components of both the acceleration theory and the neoclassical theory. Business confidence and even the technology are also interrelated. Again, as production changes only gradually from year to year, the effects of Y and Y-1 partly cancel out. Thus, such interrelated variables gives rise to the problem of multicollinearity.

To avoid the above problems, in Economic theory we often use a simple technical jargon to present the investment function in mathematical terms by considering capital stock and the time factor into consideration.

Investment has a direct relation to capital. While investment is a flow concept, capital is a stock capital. In a way, capital may be defined as cumulative net investments. Thus, we can write:

$$K_t = \sum_{i=1}^{t} I_i$$

where, K_t = capital at time t.

and I_i = net investment made during time period i.

It is to be noted that it may not be always be possible for a business firm to make all investments as planned in a particular year. Therefore, in Economics, we distinguish between actual and desired investments or stock of capital. Thus, we can write the investment function as follows:

$I_t = f(K_t^*, K_{t-1})$

Here, K_t^* means the desired level of capital., t stands for time.

[**N.B.:** Learners may consult the book entitled *"Macroeconomics: Theory and Applications"* by G. S. Gupta (Tata McGraw Hill) for detail discussion on the topic.]

CHECK YOUR PROGRESS

Q 1: Derive the expression of the demand function and define the different terms.

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Q 2: Derive the expression of the profit function and define the different terms.

....

....

....

Q 3: Identify the correct alternatives given within brackets.
Given, Q = f (L,K, R,S, V, γ). Here Q represents the functional relationship of (demand function/ profit function / investment function / production function).

3.4 LET US SUM UP

- A function is a special type of relation that expresses how one quantity depends on another quantity.
- Like mathematics, in Economics also, we notice functional relationshis, eg., demand function, supply function, profit function, utility function etc.

3.5 FURTHER READING

1) Agarwal, D.K. (2012). *Business Mathematics*, New Delhi: Vrindra Publication (p) Ltd.
2) Baruah, S. (2011). *Basic Mathematics and Its Application in Economics*, New Delhi: Trinity Press Pvt. Ltd.
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5) Kandoi, Balwant (2011). *Mathematics for Business and Economics with Application;* New Delhi: Himalaya Publishing House.

3.6 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No. 1: The demand function for a commodity can be written as:

$D_x = f\,[\,P_x, P_s, P_c, Y, T]$

where, D_x is the quantity of X demanded, P_x is the price of the commodity X, P_s denotes the price of other commodity which can be substituted for X, P_c is the price of a commodity which is a complement of commodity X, Y is the income of the consumer, T represents other demand factors such as tastes, preferences, social custom etc. and f represents a functional relationship.

Ans to Q No. 2: The profit function cab be written as:

$\pi \quad = \text{P.Q - C.....(i)}$

Here; π = Total Product,

P = Price of the product

Q = Quantity Produced

C = Total Cost of Production.

Ans to Q No. 3: Production function.

3.7 MODEL QUESTIONS

Write short notes on:

1) Demand function

2) Supply function

3) Utility function.

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UNIT 4: EQUATIONS IN ECONOMICS

UNIT STRUCTURE

4.1 LEARNING OBJECTIVES

After going through this unit, you will able to :

- understand the basic concepts of equation
- discuss how a equation is rewritten and solved
- explain the concept of demand and supply in equation
- state how cost, revenue and profit equations are framed and solved
- discuss the equation of a budget line.

4.2 INTRODUCTION

An equation represents a functional relationship or a theoritical relationship, which can be solved and value of the variables can be determined. For example, let us suppose that market supply (Q) depends on price of the commodity (P) and supply function is linear, then the supply equation can be written as Q = – c+dP where c and d are parameters. It is important to note that equations can be judged or tested. So in Economics, hypothesis about economic behaviour is represented with the help of equations.

In this unit, the concept of equation, and how it can be rewtitten and solved have been discussed in detail. Apart from it, introductory exposure to equations in Economics such as, the demand - supply equation, Cost-Revenue - Profit equation and budget line equation has been given.

4.3 CONCEPT OF EQUATION

In mathematics (and also in Economics), an equation is a statement of equality of containing one or more variables. Each side of an equation is called a member of the equation. If the two sides of an equation are true for all values of the variables involved in the equation, then the equation is called 'Identity' or 'Identical Equation'.

For example :

a) $y = ax^2 + bx + c$, is an equation of two variables (x and y), three parameters (a, b and c) and two members ('y' and '$ax^2 + bx + c$')

b) $x^2 - 16 \equiv (x + 4)(x - 4)$, is an identity as for any value of x, both the sides of the equation are equal.

Distinction between function and equation: In unit I, we have already discussed function. But there are some differences between equation and function, although both use mathematical expressions. Values of variables in the equation are solved based on the value equated, while values of variables in function are assigned. Also, equaltions always have a graph, whereas some functions can not be graphed. Most importatnly, functions are subsets of equation.

4.3.1 Degree of Equation

The degree of an equation implies the highest exponent (power) of the unknown quantity (variable) after the equation has been simplified to the rational integral form (i.e. after simplified the equation eliminating square root, negative power etc.)

For example:

(a) $x - 5 = {}^{1}/_{x}$, after simplification it becomes $x^2 - 5x - 1 = 0$ and hence degree is 2.

(b) $\sqrt{x^4 - 81} = 0$. After squaring both sides, it comes $x^4 - 81 = 0$ and hence degree is 4.

4.3.2 Polynomial Equation

The general equation of n^{th} degree in x is written as :

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n = 0.$

It is also known as polynomial equestion of n^{th} degree, as the highest exponent of x in the equation is 'n'.

If the highest exponent of x in the equation is 1, then it is known as equation of the first degree or general linear equation and it is written as $ax + b = 0 \qquad (a \neq 0)$

For example $2x - 5 = 0$

If the highest exponent of X in the equation is 2, then it is known as equation of the second degree or general quadratic equation and it is written as,

$$ax^2 + bx + c = 0 \qquad (a \neq 0)$$

For example $2 + 5x + 3x^2 = 0$

It the highest exponent of X in the equation is 3, then it is known as equation of the third degree or general cubic equation and it is written as,

$$ax^3 + bx^2 + cx + d = 0, \qquad (a \neq 0)$$

and so on.

4.3.3 Simultaneous Equation

In a set of simultaneons equations there are atleast two or more than two equations and in each equation there are two or more variables, whose value can be determined, which simultaneously satisfy all the equations. In a set of simultaneous equations number of variables should be either equal or less than the number of equations.

(a) Two variables and two equations :

$$a_{11}x_1 + a_{12}x_2 = c_1$$
$$a_{21}x_2 + a_{22}x_2 = c_2$$

For example, $3x_1 + 5x_2 = 10$

$$5x_1 + 2x_2 = 4$$

(b) Three variables and three equations :

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3$$

For example,

$$5x_1 + 2x_2 + 3x_3 = 15$$
$$5x_1 + 3x_2 + 6x_3 = 10$$
$$2x_1 + 3x_2 + 8x_3 = 12$$

(c) n variables and m equation :

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = c_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = c_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = c_m$$

4.4 REWRITING OF EQUATION

Equations are not fixed in nature. In order to make the equations simple and solvable they can be rewrite or rearranged in such a way that the equality of two sides are maintained.

For example :

(a) $x - 4 = {}^{2}/_{x}$

In order to slove or make simple the above equation is written as follows :

$x^2 - 4x = 2$

or $x^2 - 4x - 2 = 0$

Now, the equation becomes solvable.

(b) $\sqrt{x^2 + 5} = 3$

In order to solve, the above equation is written as follows :

$x^2 + 5 = 9$ (squaring both side of the equation)

$\Rightarrow x^2 + 5 = 9$

Now, the above equation becomes solvable.

4.5 SOLVING OF EQUATION

By solving of equation we simply mean to find the value of the variable of an equation that makes the equation true.

For example, the solution of the following equations are :

(a) $3x - 6 = 0$

$\Rightarrow 3x = 6$ (Rearranging the equation)

$\Rightarrow x = 6/3$

$\therefore\ x = 2$

(b) $x^2 - 25 = 0$

$\Rightarrow x^2 = 25$ (Rearranging the equation)

$\Rightarrow x = \sqrt{25}$ (Taking $\sqrt{\ }$ on both sides of the equation)

$\therefore x = \pm 5$

4.5.1 Solution of Polynomial Equation

a) Solution of 1st degree polynomial equation or general linear equation :

$ax + b = 0\ (a \neq 0)$

$\Rightarrow ax = -b$ (Rearranging the equation)

$\therefore\ x = -\left(b/a\right)$

b) Solution of 2nd degree polynomial equation or general quadratic equation :

$ax^2 + bx + c = 0, \quad (a \neq 0)$

The solution of the above equation is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

For example

$3x^2 - 4x + 1 = 0$

Here, a = 3, b = –4, c = 1

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16 - 4.3.1}}{2.3}$$

$$= \frac{4 \pm \sqrt{16 - 12}}{6}$$

$$= \frac{4 \pm \sqrt{4}}{6}$$

$$\therefore \ x = \frac{4 \pm \sqrt{4}}{6}$$

$\Rightarrow x = \dfrac{4+2}{6}$ or $\Rightarrow x = \dfrac{4-2}{6}$

$\therefore \ x = 1$ $\qquad \Rightarrow x = \dfrac{2}{6}$

$\therefore \ x = 1/3$

4.5.2 Solution of Simultaneous Equation

A set of simultaneous equations can be solved by applying any of the following method :

(1) Elimination method :

In this method we eliminate any one of the variables so as to get one equation in one variable.

Let us consider one example;

2x + 3y = 13 – (I)

x + 7y = 23 – (II)

In order to solve these two simultaneous equations (I) and (II) for finding the value of x and y, we have to eliminate either x or y. Let us eliminate x and in order to eliminate x, both sides of equation (II) is multiplied with 2, and we get 2x +14 = 46 (III)

Now, solving equation (I) and (III) we get,

$$\begin{array}{l} 2x + 3y = 13 \\ 2x + 14y = 46 \\ \underline{(-) \quad (-) \quad (-)} \\ -11y = -33 \end{array}$$

$$\Rightarrow y = \frac{33}{11}$$

$$\therefore y = 3$$

Now, putting the value of y in equation (1), we get,

$\Rightarrow$ 2x + 3.3 = 13

$\Rightarrow$ 2x + 9 = 13

$\Rightarrow$ 2x = 13–9

$\Rightarrow$ 2x = 4

$\therefore$ x = 2

$\therefore$ The solutions are x = 2 and y = 3

Verification : 2x + 3y = 2.2 + 3.3 = 13

or x + 7y = 2 + 7.3 = 23

(2) Substitution method :

In this method the value of one variable is found out from one equation and then substitute it in other equation so as to get one equation in one variable and we can solve the equation.

For example :

3x + 2y = 13 – (I)

9x – 3y = 21 – (II)

Rearranging the equation (I), we have,

3x = 13 – 2y

$$\therefore x = \frac{13 - 2y}{3} \quad -(\text{III})$$

Now, putting the value of x from equation (III) in equation (II), we get,

$$\therefore 9\left(\frac{13 - 2y}{3}\right) - 3y = 21$$

$\Rightarrow 3(13–2y) –3y = 21$

$\Rightarrow 39 – 6y – 3y = 21$

$\Rightarrow –9y = 21 – 39$

$\Rightarrow –9y = –18$

$\therefore\ y = {}^{18}\!/_{9} = 2$

Putting the value of y in equation (I), we get,

$3x + 2.2. = 13$

$\Rightarrow\ 3x + 4 = 13$

$\Rightarrow\ 3x = 9$

$\therefore\ x = 3$

$\therefore$ Solutions are x = 3 and y = 2

Varification : $3x + 2y = 3.3 + 2.2 = 13$

or $9x – 3y = 9.3 – 3.2 = 21$

(3): Cramer's method :

We will disucss this method in the next unit.

(4): Matrix method :

You will get to know this method in a higher level of study. However, the concept of matrix will be discussed in Unit 6.

CHECK YOUR PROGRESS

Q 1: What is the difference between equation and identity? (Answer in about 40 words).

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Q 2: What is a quadratic equation. (Answer in about 40 words).

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4.6 DEMAND FUNCTION

We have already discussed the concept of demand function in the previous unit in sub-section 3.3.1. Here, we shall take up certain numerical examples.

Example 4.1: The demand function of a commodity x is give by $Q_x = 15 - 3P_x$. Prepare the demand schedule, if its price varies from Rs. 5 to Rs. 1.

Solution : Given,

$Q_x = 15 - 3P_x$

The demand schedule can be obtained by putting the values of price (P) in the demand function $Q_x = 15 - 3P_x$ as

Price in (P_x)	5	4	3	2	1
Demand in units (Q_x)	0	3	6	9	12

Example 4.2: Suppose there are two consumers in the market for a good and their demand functions are as follows :

$d_1(p) = 20 - p$ for any price less than or equal to 15 and

$d_1(p) = 0$ at any price greater than 15

$d_2(p) = 30 - 2p$ for any price less than or equal to 15 and

$d_2(p) = 0$ at any price greater than 15

Find out the market demand function.

Solution : Given,

$d_1(p) = 20 - p$ [when $P \leq 15$]

$d_1(p) = 0$ [when $P > 15$]

and,

$d_2(p) = 30 - 2p$ [when $P \leq 15$]

$d_2(p) = 0$ [when $P > 15$]

The market demand function will be

d market(p) = $d_1(p) + d_2(p)$ [When $P \leq 15$]

$= 20 - p + 30 - 2p$

$= 50 - 3p$

d market (P) = 0 [When $P > 15$]

Since at p > 15, both the consumers do not want to demand the good.

Example 4.3: Suppose there are two consumers in a market and they have the similar demand function as given below.

d(p) = 40 – 4p, $p \le 10$

d(p) = 0, when $p > 10$

Find out the market demand function.

Solution : Given,

d(p) = 40 – 4p, $p \le 10$

d(p) = 0, when $p > 10$

The consumers will demand for good when price is either less than or equal to 10.

Since, there are two consumers in a market with similar demand function, therefore, the market demand function will be :

d market(p) = 2{d(p)}, when $p \le 10$

= 2 {40 – 4p}

= 80 – 8p, when $p \le 10$

d market(p) = 0, when $p > 10$

4.7 SUPPLY FUNCTION

We have already discussed the concept of supply function in the previous unit in sub-section 3.3.2. Here, we shall take up certain numerical examples.

Example 4.4: Suppose the supply function is Q_s = –10 + 2p. Calculate supply at price of Rs. 10.

Solution : Given,

Q_s = –10 + 2p

When, P = 10, then

Q_s = –10 + 2.10

= –10 + 20

= 10

∴ At P = 10, Q_s = 10 units.

Example 4.5: Suppose there are two producers in the market for a commodity and their supply functions are as follows:

$Q_1 = -10 + P$ [When $P \geq 0$]

$Q_1 = 0$ [When $P < 0$]

$Q_2 = -30 + 3P$ [When $P \geq 0$]

$Q_2 = 0$ [When $P < 0$]

Find out the market supply function.

Solution : Given,

$Q_1 = -10 + P$ [When $P \geq 0$]

$Q_1 = 0$ [When $P < 0$]

$Q_2 = -30 + 3P$ [When $P \geq 0$]

$Q_2 = 0$ [When $P < 0$]

The market supply function will be

Q market = $Q_1 + Q_2$ [When $P \geq 0$]

= (–10+P) + (–30 + 3P)

= – 40 +4P

And,

Q market = 0 [When $P < 0$]

4.8 DETERMINATION OF EQUILIBRIUM PRICE AND QUANTITY

The most common form of linear market model is

$Q_d = a - bP$ (demand function)

$Q_s = -c + dP$ (supply function)

$Q_d = Q_s$ (market clearing condition)

Where, Q_d = quantity demanded

Q_s = quantity supplied

a = intercept of demand curve

b = slope of the demand curve

c = intercept of supply curve

d = slope of supply curve

We have,

In equilibrium,

Qd = Qs

$\Rightarrow$ a – bP = –c + dP

$\Rightarrow$ bP + dP = a + c

$\Rightarrow$ P(b + d) = a + c

$$\therefore \ \overline{P} = \frac{a+c}{b+d}$$

Equilibriam quantity $\overline{Q}$ can be obtained as

$$\overline{Q} = a - b\overline{P}$$

$$= a - b\frac{a+c}{b+d}$$

$$= a - \frac{b(a+c)}{b+d}$$

$$= \frac{a(b+d) - b(a+c)}{b+d}$$

$$\therefore \ \overline{Q} = \frac{ad - bc}{b+d}$$

Example 4.6: Find equilibrium price $\left(\overline{P}\right)$ and equilibrium quantity $\left(\overline{Q}\right)$ from the following linear partial equilibrium market model.

Q_d = 20 – 7P

Q_s = –4 + 5P

Q_d = Q_s

Solution : Given,

Q_d = 20 – 7P

Q_s = – 4 + 5P

Q_d = Q_s

In equilibrium, we have

Q_d = Q_s

$\Rightarrow$ 20 – 7P = –4 + 5P

$\Rightarrow$ 12P = 24

$$\therefore \overline{P} = \frac{24}{12} = 2$$

Now substituting $\overline{P}$ = 2, in either demand function or supply function, we have

$\overline{Q}$ = 20 – 7P

= 20 – 7.2

= 20 – 14.

$\therefore \overline{Q} = 6$

4.9 COST FUNCTION

Cost function shows the functional relationship between output and cost of production. It is expressed as C = f(Q)

Where, C = cost of production, Q = quantity of output, f = functional relationship.

- **Total Cost Function:** Total cost is defined as the aggragate of all costs of producing at a given level of output. Mathematically, total cost function can be expressed as

  TC = f(Q), where Q is number of quantity produced (output)

  In short run,

  TC = ***Total Variable Cost*** + ***Total Fixed Cost***

  TC = TVC + TFC

- **Average Cost Function:** It is the cost per unit of output produced.

  $$AC = \frac{TC}{Q}$$

  Where AC = Average Cost, TC = Total Cost, Q = Quantity of output

- **Average Fixed Cost:** It is defined as the fixed cost of producing per unit of the commodity.

  $$AFC = \frac{TFC}{Q}$$

  Where, AFC = Average Fixed Cost, TFC = Total Fixed Cost, Q = Quantity of output.

- **Average Variable Cost:** It is defined as the variable cost of producing per unit of the commodity.

  $$AVC = \frac{TVC}{Q}$$

  Where, AVC = Average Variable Cost, TVC = Total Variable Cost, Q = Quantity of Output.

- **Marginal Cost Function:** Marginal cost refers to addition to total cost when one more unit of output is produced.

1) ***Total Variable Cost:*** Total variable cost refer to those costs which vary directly with the level of output. For example, wages of casual labour, payment for raw materials, power, fuel, etc.
2) ***Total Fixed Cost:*** Total Fixed Cost are costs, which do not change with the change in the quantity of output. For example, rent of factor, salary of permanent staff, insurance premium, etc. Total fixed cost is constant at all levels of output.

Symbolically,

$MC_n = TC_n - TC_{n-1}$

Where, N = Number of units produced

MC_n = Marginal cost of the n^{th} unit

TC_n = Total cost of n units

TC_{n-1} = Total cost of (n–1) units.

Mathematically, it can be derived by taking first order Derivative of TC function w.r.t. Q (output) as

$$MC = \frac{dTC}{dQ}$$

Example 4.7: The total cost function of a firm is given by

$TC = Q^3 - 6Q^2 + 2Q + 50$

Where Q is the quantity produced

Find (i) Total fixed cost

(ii) Total variable cost

(iii) Average cost

(iv) Average fixed cost

(v) Average variable cost

(vi) Marginal cost.

Solution : Given,

$TC = Q^3 - 6Q^2 + 2Q + 50$

We have,

$TC = TVC + TFC$

and, $TC = TFC$, $[\because$ when $Q = 0$, then $TVC = 0]$

Thus, at $Q = 0$

$TC = Q^3 - 6Q^2 + 2Q + 50$

$= 0^3 - 6(0)^2 + 2(0) + 50$

$\therefore$ TFC = 50

(i) Total fixed cost is 50.

(ii) $TVC = TC - TFC$ $[\because TC = TVC + TFC]$

$= Q^3 - 6Q^2 + 2Q + 50 - 50$

$\therefore TVC = Q^3 - 6Q^2 + 2Q$

(iii) We have

$$AC = \frac{TC}{Q}$$

$$= \frac{Q^3 - 6Q^2 + 2Q + 50}{Q}$$

$$\therefore AC = Q^2 - 6Q + 2 + \frac{50}{Q}$$

(iv) We have

$$AFC = \frac{TFC}{Q}$$

$$= \frac{50}{Q}$$

(v) We have

$$AVC = \frac{TVC}{Q}$$

$$= \frac{Q^3 - 6Q^2 + 2Q}{Q}$$

$$\therefore AVC = Q^2 - 6Q + 2$$

(vi) We have

$$MC = \frac{d}{dQ}TC$$

$$= \frac{d}{dQ}\left(Q^3 - 6Q^2 + 2Q + 50\right)$$

$$= \frac{d}{dQ}Q^3 - \frac{d}{dQ}6Q^2 + \frac{d}{dQ}2Q + \frac{d}{dQ}50$$

$$\therefore MC = 3Q^2 - 12Q + 2$$

[Details of differentiation will be discussed in the later chapter]

Example 4.8: The total cost of a production unit is Rs 70 and the level of output is 5 units. Find out the average cost and average variable cost if the average fixed cost is Rs 4.

Solution : Given,

TC = Rs 70

Q = 5 units

AFC = Rs 4

We have, $AC = \frac{TC}{Q} = \frac{70}{5}$ [$\because$ TC = 70, Q = 5]

$\therefore$ AC = 14.

Again, AC = AVC + AFC

$\Rightarrow$ AVC = AC – AFC

= 14 – 4 [$\because$ AFC = 4]

$\therefore$ AVC = 10.

4.10 REVENUE FUNCTION

It explains the relationship between the revenue and the quantity of a commodity demanded at a given price.

- **Total Revenue Function :** Total revenue refers to total receipts from the sale of a given quantity of a commodity. It is equal to market price (P) multiplied by the number of quantity sold of the product (Q).

  That is,

  TR = PXQ

  or, TR = ARXQ, where AR = average revenue.

  The total revenue function can be expressed as

  TR = R (Q), where Q is the number of quantity sold.

- **Average Revenue Function :**

  It is defined as total revenue per unit of output.

  Symbolically, $AR = \frac{TR}{Q}$

  Where, AR = Average revenue, TR = Total revenue, Q = Quantity sold.

- **Marginal Revenue Function :**

  It is the change in total revenue on account of the sale of an additional unit of output.

  Symbolically,

  $MR_n = TR_n - TR_{n-1}$

  Where, MR_n = Marginal revenue of the n^{th} unit

  TR_n = Total revenue of n units

  TR_{n-1} = Total revenue of (n–1) units.

Mathematically, it can be derived by taking first order derivative of TR function w.r.t. Q as

$$MR = \frac{dTR}{dQ}$$

Example 4.9: Given the total revenue function

$TR = 30 + 14Q - 17Q^2$

Find average and marginal revenue function.

Solution : Given,

$TR = 30 + 15Q - 17Q^2$

We have,

$$AR = \frac{TR}{Q}$$

$$= \frac{30 + 15Q - 17Q^2}{Q}$$

$$= \frac{30}{Q} + 15 - 17Q$$

Again,

$$MR = \frac{dTR}{dQ}$$

$$= \frac{d}{dQ}\left(30 + 15Q - 17Q^2\right)$$

$= 15 - 34Q$

[Details of differentiation will be discussed in the later chapter]

4.11 PROFIT FUNCTION

Profit (denoted by the symbol π) is the difference between total revenue (TR) and total cost (TC). That is,

$\pi = TR - TC$

Where, π = profit function.

Example 4.10: In a perfectly competitive market, the total revenue and total cost of a firm are given by :

$TR = 20Q$

$TC = Q^2 + 4Q + 20$

Find maximum profit if Q = 8

Solution : Given,

$$TR = 20Q$$

$$TC = Q^2 + 4Q + 20$$

We have,

$$\begin{aligned}\pi &= TR - TC \\ &= 20Q - (Q^2 + 4Q + 20) \\ &= 20Q - Q^2 - 4Q - 20\end{aligned}$$

at Q =8, $\pi = 20.8 - 8^2 - 4.8 - 20$

$$\begin{aligned}&= 160 - 64 - 32 - 20 \\ &= 160 - 116\end{aligned}$$

$$\therefore\ \pi = 44$$

Example 4.11: A monopolist has the following average revenue (AR) and total cost (TC) functions

$$AR = 30 - Q$$

$$TC = Q^3 - 15Q^2 + 10Q + 100$$

Find maximum profit, when Q = 10

Solution : Given,

$$AR = 30 - Q$$

$$\therefore\ TR = AR.Q = (30–Q)Q = 30Q - Q^2$$

and $\quad TC = Q^3 - 15Q^2 + 10Q + 100$

We have,

$$\begin{aligned}\pi &= TR - TC \\ &= (30Q - Q^2)–(Q^3 - 15Q^2 + 10Q + 100) \\ &= 30Q - Q^2 - Q^3 + 15Q^2 - 10Q - 100 \\ &=30.10 - (10)^2 –(10)^3 + 15.(10)^2 - 10.10 –100 \\ &= 300 - 100 - 1000 + 1500 - 100 - 100 \\ &= 1800 - 1300\end{aligned}$$

$$\therefore\ \pi = 500$$

4.12 BUDGET FUNCTION

Budget line is a graphical representation of all possible combination of two goods which can be purchased with given income and prices, such that the cost of each of these combinations is equal to the money income of the consumer. In other wards, the budget line shows different combinations of two goods which a consumer can attain at given income and market price of the goods.

The budget line can be expressed as an equation :

$M = P_xQ_x + P_yQ_y$

Where M = Money expenditure

P_x = price of good x

P_y = price of good y

Q_x = Quantity of good x

Q_y = Quantity of good y

Example 4.12: The total money income of a consumer is M and he spends his entire money income on the consumption of two commodities X and Y. The price of X and Y are P_x and P_y respectively. State the budget equation.

Solution : Given,

Money income of the consumer = M

Price of commodity X = Px

Price of commodity Y = P_y

Quantity of commodity X = Q_x

Quantity of commodity Y = Q_y

$\therefore$ The budget equation of the consumer is

$M = P_xQ_x + P_yQ_y$

Example 4.13: A consumer with his given money income can buy 10 units of the commodity x and 15 units of the commodity y and the price of X and Y are Rs 12 and Rs 10 respectively. Find out the income level of the consumer.

Solution : Given,

Q_x = 10 units

Q_y = 15 units

P_x = Rs. 12

P_y = Rs. 10

$\therefore$ The income level of the consumer will be.

$M = P_xQ_x + P_yQ_y$

$= 12.10 + 10.15$

$= 120 + 150$

$= 270$

$\therefore$ M = Rs. 270

CHECK YOUR PROGRESS

Q 3: Suppose the demand function is $Q_d = 40 - 5P$. Calculate

(i) demand at price Rs 2.

(ii) price, when demand will be 0.

(iii) demand, when price will be 0.

Q 4: Given the market model

$Q_d = 18 - 2P$

$Q_s = -7 + 3P$

$Q_d = Q_s$

Obtain the equilibrium price and quantity.

Q 5: A firm has the following price and total cost function

$TR = 100Q - 4Q^2$

$TC = 50 + 20Q$

Find the profit function.

4.13 LET US SUM UP

- Demand function shows the relationship between demand for a commodity and its various determinants.
- Supply function refers to the functional relationship between supply of a commodity and its various determinants.
- Calculation of different costs

TC = TFC + TVC or ACXQ

TVC = TC – TFC or AVCXQ

TFC = TC – TVC or AFCXQ

$AC = \frac{TC}{Q}$ or AFC + AVC

$AVC = \frac{TVC}{Q}$ or AC – AFC

$AFC = \frac{TFC}{Q}$ or AC – AVC

$MC_n = TC_n - TC_{n-1}$, or $\frac{d}{dQ}TC$

- Calculation of different revenues

  TR = PXQ or ARXQ

  $AR = \frac{TR}{Q}$

  $MR_n = TR_n$ – TRn-1 or $\frac{d}{dQ}TR$

- Profit is the difference between revenue and cost.

  π = TR – TC

- Budget line is locus of different combinations of two goods which a consumer can attaing at given income and market price of the goods.

4.14 FURTHER READING

1) Agarwal, D.K. (2012). *Business Mathematics*, New Delhi: Vrindra Publication (p) Ltd.
2) Baruah, S. (2011). *Basic Mathematics and Its Application in Economics*, New Delhi: Trinity Press Pvt. Ltd.
3) Bose, D. (2004). *Mathematical Economics;* New Delhi: Himalaya Publishing House.
4) Chiang, A.C. (2006) *Fundamental Methods of Economics Analysis*; New Delhi: MC Graw Hill Education India.
5) Kandoi, Balwant (2011). *Mathematics for Business and Economics with Application;* New Delhi: Himalaya Publishing House.

4.15 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: If the two sides of an equation are true for all values of the variables involved in the equation, than the equation is called identity.

Ans to Q No 2: It the highest exponent of the independant variable in a polymial equation is 2, than it is known as quadratic equation and it takes the form of $ax^2 + bx + c = 0 \quad (a \neq 0)$

Ans to Q No 3: Given, $Q_d = 40 - 5p$

(i) $Q_d = 40 - 5.2 \qquad [\because P = 2]$

$= 40 - 10$

$= 30$

$\therefore Q_d = 30$ units, when P = Rs. 2

(ii) $Q_d = 40 - 5P$

$\Rightarrow 0 = 40 - 5P \quad [\because Q_d = 0]$

$\Rightarrow 5P = 40$

$\Rightarrow P = 8$

$\therefore$ P = Rs. 8, when $Q_d = 0$

(iii) $Q_d = 40 - 5P$

$\Rightarrow Q_d = 40 - 5.0 \qquad [\because P = 0]$

$\Rightarrow Q_d = 40$

$\therefore Q_d = 40$ units, when P = 0.

Ans to Q No 4: $Q_d = 18 - 2P$

$Q_s = -7 + 3P$

$Q_d = Q_s$

In equilibrium, we have.

$Q_d = Q_s$

$\Rightarrow 18 - 2P = -7 + 3P$

$\Rightarrow 5P = 25$

$\Rightarrow \overline{P} = 5$

Now substituting $\overline{P} = 5$, in either demend function or supply function, we have

$\overline{Q} = 18 - 2P$

$= 18 - 2 \times 5$

$\therefore \overline{Q} = 8$

Ans to Q No 5: Given,

TR = 100Q – 4Q^2

TC = 50 + 20Q

The profit function of the firm will be

π = TR – TC

= (100Q – 4Q^2) – (50 + 20Q)

= 100Q – 4Q^2 – 50 – 20Q

π = 80Q – 4Q^2 – 50.

4.16 MODEL QUESTIONS

Q 1: Suppose there are two consumers in a market. The demand functions are as follows :

$d_1(P) = 25 - P$ [when $P \leq 25$]

$d_1(P) = 0$ [when p>25]

and $d_2(P) = 20 - 2P$ [when $P \leq 25$]

$d_2(P) = 0$ [when p>25]

Find out the market demand function.

Q 2: Given the supply function

$Q_s = -14 + 2P$

Calculate (a) supply at price Rs. 10

(b) at what price, supply will be 0

(c) the price at which firm will be willing to supply 15 units.

Q 3: Derive the equilibrium price and equilibrium quantity of the following market model.

$Q_d = 20 - 2.5P$

$Q_3 = -4 + 1.5P$

$Q_d = Q_3$

Q 4: Give the short-run average cost function

$$AC = 2q^2 - 15Q + 30 + \frac{16}{Q}$$

Find (a) TC

(b) AVC

(c) AFC

(d) MC

Q 5: Solve the following equations :

(a) $2x + 3y = 9$

$3x + 5y = 12$

(b) $5x_1 + 7x_2 = 85$

$3x_1 + 9x_2 = 75$

(c) $7x_1 - 2x_2 = 8$

$3x_1 + 6x_2 = 24$

(d) $4x - 3y = 11$

$2x + 2y = 16$

*** ***** ***

UNIT 5: LIMIT AND CONTINUITY

UNIT STRUCTURE

5.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define the concept of limit.
- evaluate limits by algebraic rules.
- explain the concept of continuity of a function.
- show that some well known functions are continuous.

5.2 INTRODUCTION

So far, we have discussed the Set theory and functions.

In this unit, we shall discuss one of the fundamental concepts in Mathematics, which is of paramount importance to the study of differential calculus. This concept is known as 'Limit'.

The concept of limit is an abstract one. We will learn various algebraic rules to find limits at various points of well-known functions. We will also learn about left and right hand limits. We will also discuss about the continuity and differentiation of functions. Finally, we will discuss about the concept of integration. We will also discuss about two powerful methods of integration, namely, methods of substitution and the method of integration by parts.

5.3 LIMITS

We are already familiar with the concept of a real function and a real variable. The concept of limit is an abstract one. Consider the function $f(x) = \dfrac{x^2 - 9}{x - 3}$, which is defined at all points except $x = 3$. In the first column of following table, some values of x close to 3 are written and in the second column the corresponding values of $f(x)$ are written.

The value of the variable	The corresponding value of $f(x) = \dfrac{x^2-9}{x-3}$
2.97	5.97
2.98	5.98
2.99	5.99
3.01	6.01
3.02	6.02
3.03	6.03

From the above table, we observe that if x is very close to 3, then $\dfrac{x^2-9}{x-3}$ is very close to 6. It is also true that for all values of x close to 3, $\dfrac{x^2-9}{x-3}$ is close to 6. We then say:

As x tends to 3, the function $\dfrac{x^2-9}{x-3}$ tends to 6 or equivalently, we say: The function $\dfrac{x^2-9}{x-3}$ has the limit 6 at the point 3.

In symbols, we write

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=6.$$

In general, by $\underset{\substack{x\to a\\ |x-a|x\to a}}{\text{Lim}} f(x)=1$ we mean that as gets close to *a* then $f(x)$ gets close to a limit of l. This can be put in a more precise way as: $|f(x)-l|$ can be made arbitrarily small, by taking $|x-a|$ to be sufficiently small.

> The value of the function *f* at *a* is immaterial for finding the limit of *f* at *a*. Only the values that are close to *a* but not at *a* matter.

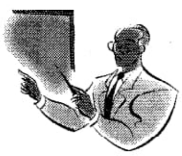
The following two results are simple consequences of the definition.

1. **Constant Rule:** $\lim_{x\to a} c = c$, that is, the limit of the constant function $f(x)=c$ as x tends to a is the value c.

2. **Identity Rule:** $\lim_{x\to a} x = a$, that is, the limit of the identity function $f(x)=x$ as x tends to a is the value a.

Uniqueness of Limit

> If $\lim_{x\to a} f(x)$ exists, it is unique. There cannot be two distinct numbers l_1 and l_2 such that when x tends to a the function $f(x)$ tends to both l_1 and l_2.



LET US KNOW

(1) The symbol $x\to a$ (read as x tends to a) means that 'x' assumes successive values one after another either from the right or from the left and approaches very near to a, so that $|x-a|$ becomes almost zero.

(2) Distinction between f(a) and $\lim_{x\to a} f(x)$:

(i) f(a) stands for the value of f(x) when x is exactly equal to a ... f (a) may be otbained either by definition of the function f(x) at x=a or by putting x=a in f(x) when it exists.

(ii) f(a) and $\lim_{x\to a} f(x)$ may be equal or unequal when both exist.

(iii) both of f(a) and $\lim_{x\to a} f(x)$ may exist or one of them may exist and other may not exist.

5.3.1 Finding Limits Analytically

It is possible to find limits by using algebraic techniques. The limit behaves well with respect to the operations of addition, subtraction, multiplication by a constant, multiplication, division, etc. The most basic theorem governing analysis of limits is the Principal Limit Theorem.

Principal Limit Theorem:

Let *a* and *k* be real numbers, $n \geq 0$, and let *f* and g be functions with limits at *a* such that $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$. Then

1. **Sum Rule:** $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = l + m.$

2. **Difference Rule:**

$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = l - m.$

3. **Product Rule:** $\lim_{x\to a}[f(x)\bullet g(x)] = \lim_{x\to a} f(x) \bullet \lim_{x\to a} g(x) = l \bullet m.$

4. **Quotient Rule:** $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{l}{m}$ provided

$\lim_{x\to a} g(x) = m \neq 0.$

5. **Coefficient Rule:** $\lim_{x\to a} k\, f(x) = k \lim_{x\to a} f(x) = kl.$

6. **Power Rule:** $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = l^n.$

We can state the above rules in language. For example, Sum Rule could be expressed as **"The limit of a sum is the sum of the limits."**

5.3.2 Examples of Evaluation of Limits Using Various Rules

Example 5.1: Find $\lim_{x\to 2} x^2$.

Solution: $\lim_{x\to 2} x^2 = \left(\lim_{x\to 2} x\right)^2$ (by Power Rule)

$= 2^2$ (by Identity Rule)

$= 4$

Example 5.2: Find $\lim_{x\to a} 2x^3+3x^2+4x+5$.

Solution:

$$\lim_{x\to a} 2x^3+3x^2+4x+5$$

$= 2\lim_{x\to a} x^3 + 3\lim_{x\to a} x^2 + 4\lim_{x\to a} x + \lim_{x\to a} 5$ (by Sum and Coefficient Rules)

$= 2\left(\lim_{x\to a} x\right)^3 + 3\left(\lim_{x\to a} x\right)^2 + 4\lim_{x\to a} x + \lim_{x\to a} 5$ (by Power Rule)

$= 2a^3+3a^2+4a+5.$ (by Identity and Constant Rules)

Remark: It is clear from the above example that for all polynomial functions, the limit at a point can be obtained by substituting that value for the variable *x*. But, for non-polynomial functions, this may not be true always. The next two examples show this fact.

Example 5.3: Find $\lim_{x\to 1} \frac{2x-1}{x^3-2}$.

Solution:

$$\lim_{x\to 1} \frac{2x-1}{x^3-2} = \frac{\lim_{x\to 1}(2x-1)}{\lim_{x\to 1}(x^3-2} \quad \text{(by Quotient Rule)}$$

$$= \frac{\lim_{x\to 1} 2x - \lim_{x\to 1} 1}{\lim_{x\to 1} x^3 - \lim_{x\to 1} 2} \quad \text{(by Difference Rule)}$$

$$= \frac{2\lim_{x\to 1} x - \lim_{x\to 1} 1}{(\lim_{x\to 1} x)^3 - \lim_{x\to 1} 2} \quad \text{(by Coefficient and Power Rules)}$$

$$= \frac{2(1)-1}{(1)^3-2} \quad \text{(by Constant and Identity Rules)}$$

$= -1.$

Example 5.4: Find $\lim_{x\to 2} \frac{x^2-4}{x^2-3x+2}$.

Solution: Since the denominator function x^2-3x+2 has the limit 0 at the point 1, that is, $\lim_{x\to 2} x^2-3x+2=0$, so Quotient Rule cannot be applied here. Therefore, we have to resort to another method. Factorizing both the numerator and the denominator, we find that

$$\frac{x^2-4}{x^2-3x+2}=\frac{(x-2)(x+2)}{(x-1)(x-2)}.$$

When $x \neq 2$, we can cancel the common factor $x - 2$.

Thus,

$$\frac{x^2-4}{x^2-3x+2}=\frac{x+2}{x-1}.$$

Therefore,

$$\lim_{x\to 2}\frac{x^2-4}{x^2-3x+2} = \lim_{x\to 2}\frac{x+2}{x-1} = \frac{\lim\limits_{x\to 2} x + \lim\limits_{x\to 2} 2}{\lim\limits_{x\to 2} x - \lim\limits_{x\to 2} 1} = \frac{2+2}{2-1} = 3$$

CHECK YOUR PROGRESS

Q 1: Write true (T) or false (F).

(a) $\lim\limits_{x\to 1}(x-1)^2+5=5.$ (T / F)

(b) $\lim\limits_{x\to 2}(x-2)(x+4)=0.$ (T/F)

(c) $\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}=0.$ (T / F)

(d) $\lim\limits_{x\to -1}\dfrac{x^2-1}{x+1}=2.$ (T/F)

5.3.3 Limitis by the Method of Substitution

Consider the example $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}$. We find that

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}=\lim_{x\to 1}(x+1)=2$$

The same limit can be evaluated with the help of substitution. If we substitute $y=x-1$, i.e. $x=y+1$, in the expression $\dfrac{x^2-1}{x-1}$, then $\dfrac{x^2-1}{x-1}=\dfrac{(y+1)^2-1}{y}=y+2$. Also, note that when x tends 1, then y

tends 0. Therefore,

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{y \to 0}(y + 2) = 2.$$

In general,

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a + h).$$

Equivalently,

$$\lim_{x \to a} f(x) = \lim_{y \to a+h} f(y), \quad \textit{where } y = x + h.$$

5.3.4 A Special Limit

Let us evaluate $\lim_{x \to a} \frac{x^n - a^n}{x - a}$, where n is a positive integer.

By the method of substitution, we find that

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{h \to 0} \frac{(a + h)^n - a^n}{a + h - a}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[a^n + \binom{n}{1}a^{n-1}h + \binom{n}{2}a^{n-2}h^2 + \ldots + h^n - a^n\right]$$

[By Binomial Theorem]

$$= \lim_{h \to 0}\left[\binom{n}{1}a^{n-1} + \binom{n}{2}a^{n-2}h + \ldots + h^{n-1}\right] = \binom{n}{1}a^{n-1} = na^{n-1}.$$

Thus,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}, \text{ where } n \text{ is a positive integer.}$$

The above formula remains true, when n is a rational number, provided a is positive.

Example 5.5: Find $\lim_{x \to 3} \frac{x^3 - 27}{x^2 - 9}$.

Solution:

$$\lim_{x \to 3} \frac{x^3 - 27}{x^2 - 9} = \lim_{x \to 3}\left[\frac{x^3 - 27}{x - 3} \div \frac{x^2 - 9}{x - 3}\right] = \lim_{x \to 3}\left[\frac{x^3 - 27}{x - 3}\right] \div \lim_{x \to 3}\left[\frac{x^2 - 9}{x - 3}\right]$$

$$= \lim_{x\to 3}\left[\frac{x^3-3^3}{x-3}\right] \div \lim_{x\to 3}\left[\frac{x^2-3^2}{x-3}\right] = 3(3)^2 \div 2(3) = 27 \div 6 = \frac{9}{2}.$$

Example 5.6: If $\lim_{x\to 2}\frac{x^n-2^n}{x-2}=12$ and if n is a positive integer, find n.

Solution: We have $\lim_{x\to 2}\frac{x^n-2^n}{x-2}=n2^{n-1}$. According to the given question, $n2^{n-1}=12=3.2^2$. Therefore, n = 3.

5.3.5 Some Other Special Limits

The following limit rule exponential functions are very useful.

Example 5.7: Find $\lim_{x\to 0}\frac{e^{4x}-1}{x}$.

Solution: We have

$$\lim_{x\to 0}\frac{e^{4x}-1}{x} = \lim_{x\to 0}\frac{e^{4x}-1}{4x}\cdot 4 = 4\lim_{y\to 0}\frac{e^y-1}{y} = 4.$$

5.3.6 One Sided Limits

In this section we will discuss one sided limits.

The notation $\lim_{x\to a+} f(x)=l$ means that when is close x to a and greater than a then $f(x)$ is close to l.. Similarly, $\lim_{x\to a-} f(x)=l$ means that when is close x to a and less than a, then $f(x)$ is close to l.

$\lim_{x\to a+} f(x)$ is called the **right limit** of f at a and $\lim_{x\to a-} f(x)$ is called the **left limit** of f at a. Both these are called **one-sided limits** of f at a

Example 5.8: Find $\lim_{x\to 0+}\frac{x}{|x|}$ and $\lim_{x\to 0-}\frac{x}{|x|}$.

Solutions: When x takes positive values, then $\frac{x}{|x|}=\frac{x}{x}=1$.

Therefore, we have $\lim_{x\to 0+}\frac{x}{|x|}=1$.

On the other hand, when takes negative values, then

$\frac{x}{|x|} = \frac{x}{-x} = -1$. So, $\lim\limits_{x \to 0-} \frac{x}{|x|} = -1$.

If $\lim\limits_{x \to a+} f(x) \neq \lim\limits_{x \to a-} f(x)$ then $\lim\limits_{x \to a} f(x)$ does not exist.

If $\lim\limits_{x \to a+} f(x) = \lim\limits_{x \to a-} f(x) = l$ then $\lim\limits_{x \to a} f(x)$ exists and $\lim\limits_{x \to a} f(x) = l$.

Example 5.9: Decide whether $\lim\limits_{x \to 0} \frac{x}{|x|}$ exists or not.

Solution: From the solutions of Example 11, we note that

$\lim\limits_{x \to 0+} \frac{x}{|x|} = 1$ and $\lim\limits_{x \to 0-} \frac{x}{|x|} = -1$. Since the right limit and left limit are not equal, therefore $\lim\limits_{x \to 0} \frac{x}{|x|}$ does not exist.

CHECK YOUR PROGRESS

Q 2: Write true (T) or false (F).

(a) If *f* is a real function such that *f* (*x*) = *f* (–*x*) for all *x* , then $\lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to 0-} f(x)$. (T /F)

(b) $\lim\limits_{x \to 1} [x] = 1$. (T / F)

(c) $\lim\limits_{x \to 1+} [x] = 1$. (T / F)

Q 3: Find the following limits:

(i) $\lim\limits_{x \to 2} \frac{x-2}{x+2}$ (ii) $\lim\limits_{x \to 0} \frac{(1+x)^{n}-1}{(1+x)^2-1}$ (iii) $\lim\limits_{x \to 3} \frac{e^{x}-e^{3}}{x-3}$

5.4 CONTINUITY

In this section, we will learn about the concept of continuity of a function. We will show that a large class of well known functions is continuous.

5.4.1 Basic Definition and Examples

Let *f* be a real function and let *a* be in the domain of *f* If the limit of *f* at *a* is same as the value of *f* at *a*, then *f* is said to be continuous at *a*.

Thus, if *f* is to be continuous at *a* the following three conditions must be satisfied.

a) *f (a)* is defined,

b) $\lim_{x \to a} f(x)$ exists,

c) $\lim_{x \to a} f(x) = f(a)$.

Example 5.10: Show that every constant function is continuous at all points.

Solution: Consider a constant function $f(x) = c$ and an arbitrary point a.

Then $f(a) = c$ and $\lim_{x \to a} f(x) = \lim_{x \to a} c = c$, and so, $\lim_{x \to a} f(x) = f(a)$.

Therefore, f is continuous at a

Example 5.11: Show that the identity function is continuous at all points.

Solution: Consider the identity function $f(x) = x$ and an arbitrary point *a*.

Then, we have $f(a) = a$ and $\lim_{x \to a} f(x) = \lim_{x \to a} x = a$.

Thus, $\lim_{x \to a} f(x) = f(a)$, which shows that f is continuous at 'a'.

Example 5.12: Show that the greatest integer function [*x*] is not continuous at 0.

Solution: Let $f(x) = [x]$.

Now, $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} [x] = 0$ and $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} [x] = -1$.

Thus, the right hand and left hand limits of f at 0 are not equal. Which means that $\lim_{x \to 0} f(x) = \lim_{x \to 0} [x]$ does not exist. Therefore, though $f(0) = [0] = 0$, *f* is not continuous at 0.

Some more results are listed in the following table.

(a) The function x^2 is continuous at 3.

(b) The function $\dfrac{x-4}{x-1}$ is continuous at 2.

(c) The function e^x is continuous at 0.

(d) The function $\frac{x}{|x|}$ is not continuous at 0.

Definition:

A real function is said to be continuous in an open or closed interval if it is continuous at every point of the interval.

Remark: When a function f is considered on a closed interval $[a,b]$, then f is said to be continuous at the end point a if $\lim_{x \to a+} f(x) = f(a)$. Similarly, f is said to be continuous at the end point b if $\lim_{x \to b-} f(x) = f(b)$.

5.5 LET US SUM UP

- In this unit, we have learnt about the concept of limits and various algebraic rules to find limits at various points of well-known functions. We have also learnt about the left and right hand limits, and found that they need not be equal always.
- We have learnt about the continuity of functions. In summary, we can say that if the limit exists at a point and it is equal to the value of the function at that point then the function is continuous at that point.

5.6 FURTHER READING

1) Agarwal, D.K. (2012). *Business Mathematics*, New Delhi: Vrindra Publication (p) Ltd.
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3) Bose, D. (2004). *Mathematical Economics*; New Delhi: Himalaya Publishing House.
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5.7 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: (a) T, (b) T, (c) F, (d) F

Ans to Q No 2: (a) T, (b) F, (c) T.

Ans to Q No 3: (i) 0 (ii) 3, (iii) e^3.

5.8 MODEL QUESTIONS

Q 1: Evaluate the following limits:

(a) $\lim\limits_{x \to 3} \dfrac{x^2 - 7x + 12}{x^2 - 4x + 3}$ (b) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x} - 1}{x}$

(c) $\lim\limits_{x \to 0} \dfrac{\sin 10x}{\sin 11x}$ (d) $\lim\limits_{x \to k} \dfrac{x^{10} - k^{10}}{x^5 - k^5}$

Q 2: Suppose a function *f* is defined as follows.

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x > 0. \end{cases}$$

By evaluating the left and right hand limits, show that $\lim\limits_{x \to 0} f(x) = 0$.

Q 3: If *f* is an odd function (i.e., $f(-x) = -f(x), \quad \forall x.$) and $\lim\limits_{x \to 0} f(x)$ exists, then show that the value of this limit is zero.

Q 4: Decide whether the function given in Question 2 is continuous at $x = 0$ or not.

Q 5: A real function *f* is defined as follows.

$$f(x) = \begin{cases} \dfrac{\tan x}{x}, & x \neq 0 \\ \dfrac{3}{4}, & x = 0. \end{cases}$$

Show that *f* is continuous at 0.

*** ***** ***

UNIT 6 : DIFFERENTIATION

UNIT STRUCTURE

6.1 LEARNING OBJECTIVES

After going through this unit, you will be able to :

- understand the geometrical meaning of $f'(x)$ or $\frac{dy}{dx}$
- know the different rules of differentiation

6.2 INTRODUCTION

This unit introduces you to the concept of derivative. After going through this unit, you will be able to know the different rules of differentation.

6.3 CONCEPT OF DERIVATIVE OF A FUNCTION

Many phenomena involve changing quantities and their rates of change (increase or decrease). Differential calculus deals with the rate of change of a dependent variable with respect to an independent variable.

Let f be a real valued function and a is a point in its domain of definition.

Then the derivative of f at a is defined by

$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ provided this limit exists and is donoted by $f'(a)$.

If y = f(x) be a function of f, the derivative of y with respect to x is $\frac{dy}{dx}$ or y_1.

When both x and y increase or decrease simulteneously, the derivative $\frac{dy}{dx}$ is positive. But when y increases with a decrease in x, or decreases with an increase in x, then $\frac{dy}{dx}$ is negative.

Remark : The derivative of a function of x is also called as the differential coefficient of x. The process of finding the differential coefficient is called differentiation.

Example 6.1 **:** Find the derivative of f(x) = 3x at x = 2

Solution : We have

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

$$\therefore f'(2) = \lim_{h \to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h \to 0} \frac{3(2+h)-3(2)}{h}$$

$$= \lim_{h \to 0} \frac{6+3h-6}{h}$$

$$= \lim_{h \to 0} 3$$

$$= 3$$

$\therefore$ Derivative of the function f(x) = 3x at x = 2 is 3.

Example 6.2 : If f(x) = $2x^2 + 3x - 5$, then using definition show that $f'(0) + 3f'(-1) = 0$.

Solution : Given f(x) = $2x^2 + 3x - 5$

We have

$$f'(0) = \lim_{h \to 0} \frac{f(o+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{(3h^2 + 3h - 5) - (0 + 0 - 5)}{h}$$

$$= \lim_{h \to 0} \frac{h(3h+3)}{h}$$

$$= \lim_{h \to 0} (3h + 3) \qquad [\because h \neq 0 \; as \; h \to 0]$$

$$= 3$$

Also $f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$

$$= \lim_{h \to 0} \frac{\{2(-1+h)^2 + 3(-1+h) - 5\} - \{2(-1)^2 + 3(-1) - 5\}}{h}$$

$$= \lim_{h \to 0} \frac{2 - 4h + 2h^2 - 3 + 3h - 5 - (2 - 3 - 5)}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 - h}{h}$$

$$= \lim_{h \to 0} \frac{h(2h - 1)}{h}$$

$$= \lim_{h \to 0} (2h - 1) = -1$$

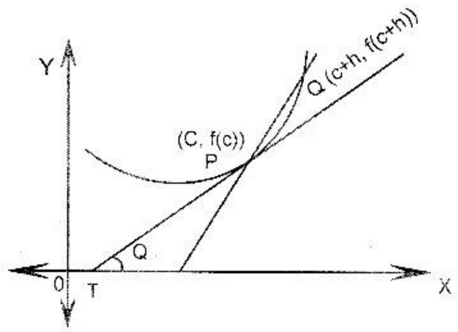
$\therefore f'(0) + 3f'(-1) = 0$

6.4 GEOMETRICAL INTERPRETATION OF DERIVATIVE

Let y=f(x) be a function. Let P (c, f(c)) and Q (c+h, f(c+h)) be two neighbouring points on the graph of y=f(x).

Slope of Chord PQ $= \dfrac{f(c+h)-f(c)}{(c+h)-c}$

$= \dfrac{f(c+h)-f(c)}{h}$



When $Q \to P$ along the curve, the chord PQ becomes tangent PT at P. Thus, the slope of the chord PQ becomes slope of the tangent at P when $Q \to P$, i.e. $h \to 0$

$\therefore$ Slope of the tangent at P$= \lim\limits_{h \to 0} \dfrac{f(c+h)-f(c)}{h}$

$= f'(c)$ (by definition)

Note :- If tangent is parallel to x-axis, then $f'(c)=0$.

6.5 ECONOMIC INTERPRETATION OF DERIVATIVE

Derivative normally represents the marginal function. In case of a simple cost function y = f(x) where y represents the total cost of production and x denotes the output, an infinitely small increase in output (Δx) leads to a corresponding infinitely small increase in cost (Δy). The derivative of a cost function $\lim\limits_{x \to 0} \dfrac{\Delta y}{\Delta x}$ at point x is the marginal cost at the output level x.

6.6 RULES OF DIFFERENTIATION

In this section, we shall discuss the rules of differentiation.

6.6.1 The Rules for Sums and Differences

Theorem 1: If f_1 and f_2 are two differentiable functions on the interval I, and f be their sum function so that $f = f_1 + f_2$. Then f is differentiable and

$$\frac{d}{dx}f = \frac{d}{dx}f_1 + \frac{d}{dx}f_2$$

Theorem 2 If f_1 and f_2 are two differentiable functions on the interval I, and f be their difference function so that $f=f_1-f_2$ then f is differentiable

and $\frac{d}{dx}f = \frac{d}{dx}f_1 - \frac{d}{dx}f_2$

Illustrative Examples:

Example 6.3 **:** Find the derivatives of the following with respect to x.

a) $x(1+x)^2$ b) $b\log_e x+x^3+\sqrt{x}$

c) $4\log_{10}x$

Solution :

a) $\frac{d}{dx}\{x(1+x)^2\}=\frac{d}{dx}\{x(1+2x+x^2)\}$

$$= \frac{d}{dx}(x+2x^2+x^3)$$

$$= \frac{d}{dx}(x)+2\frac{d}{dx}(x^2)+\frac{d}{dx}(x^3)$$

$$= 1 + 2(2x) + 3x^2$$

$$= 1 + 4x + 3x^2$$

b) $\frac{d}{dx}\left(b\log_e x + x^3 + \sqrt{x}\right)= b\frac{d}{dx}(\log_e x)+\frac{d}{dx}(x^3)+\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$

$$= b.\frac{1}{x}+3x^2+\frac{1}{2}x^{\frac{1}{2}-1}$$

$$= \frac{b}{x}+3x^2+\frac{1}{2\sqrt{x}}$$

c) $\frac{d}{dx}(4\log_{10}x)=\frac{d}{dx}\left(\frac{4\log_e x}{\log_e 10}\right)$

$$= \frac{4}{\log_e 10}\frac{d}{dx}(\log_e x)$$

$$= 4\log_{10}e\frac{1}{x}$$

$$= \frac{4\log_{10}e}{x}$$

6.6.2 Product Rule of Differentiation

Theorem 3 : Let u(x) and v(x) be two differentiable functions. Then the product u(x) v(x) is also differentiable and

$$\frac{d}{dx}\big(u(x)v(x)\big) = u(x)v'(x) + v(x)u'(x)$$

Corollary : If u, v, w are three differentable functions , then

$$\frac{d}{dx}(u(x)v(x)w(x)) = u'(x)v(x)w(x) + v'(x)u(x)w(x) + w'(x)u(x)v(x)$$

Theorem 4: Let u(x) and v(x) be two differentiable functions in the same interval, then the quotient function $\frac{u(x)}{v(x)}$ is also differentiable and

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{v(x)\frac{d}{dx}(u(x)) - u(x)\frac{d}{dx}(v(x))}{(v(x))^2}$$

$$= \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}$$

Example 6.4 : Find the derivatives of the following functions with respect to x.

a) x^2e^x b) $e^x\log x$

Solution : a) $\frac{d}{dx}(x^2e^x) = e^x\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(e^x)$

$= e^x.2x + x^2.e^x$

$= xe^x(2+x)$

b) $\frac{d}{dx}(e^x\log x) = \log x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(\log x)$

$= \log x.e^x + e^x.\frac{1}{x}$

$= e^x\left(\log x + \frac{1}{x}\right)$

6.6.3 Derivative of Composite Functions or Chain Rule of Differentiation

If $y = \phi(t)$ and $t = f(x)$, then $y = \phi(f(x))$ is a composite function.

Theorem 5: Let y = ϕ (t) when t=f(x). Then

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$$

Generalisation of above theorem :

Let y=f(u), u=g(t), t=h(x) then

$$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dt}.\frac{dt}{dx}$$

Thus, if the composite function y is a function of many functions, then

$$\frac{dy}{dx}=\frac{d(1^{st}\ \text{function})}{d(2^{nd}\ \text{function})}.\frac{d(2^{nd}\ \text{function})}{d(3^{rd}\ \text{function})}\cdots\frac{d(\text{last function})}{dx}$$

Above is also known as **Chain Rule of Differentiation**.

Example 6.5 : Find the derivatives of the following functions with respect to x.

$y = \log(\sqrt{x}+\frac{1}{\sqrt{x}})$

Solution : Let t = $\sqrt{x}+\frac{1}{\sqrt{x}}$

$\therefore$ y=log t

By chain rule,

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$$

$$=\frac{d}{dt}(\log t).\frac{d}{dx}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$$

$$=\frac{1}{t}\left(\frac{1}{2\sqrt{x}}-\frac{1}{2x\sqrt{x}}\right)$$

$$=\frac{1}{\sqrt{x}+\frac{1}{\sqrt{x}}}.\frac{x-1}{2x\sqrt{x}}$$

$$=\frac{\sqrt{x}}{x+1}.\frac{x-1}{2x\sqrt{x}}$$

$$=\frac{x-1}{2x(x+1)}$$

6.7 SECOND ORDER DERIVATIVE

As we have already said earlier, the deriviative $\frac{dy}{dx}$ (or, $f'(x)$) of a function y = f(x) is also a function of x. If the funciton $f(x)$ is differentiable, the function $f'(x)$ can be differentiated with respect to x. The derivative obtained by differentiating $f'(x)$ is called the second order derivative of a function y = (x) andis written as:

$$\frac{d}{dx}[f'(x)] = f''(x) = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d^2y}{dx^2}.$$

Example 6.6:

If $y = f(x) = 5x^3 - 3x^2 + 2x + 10$

$$\frac{dy}{dx} = \frac{d}{dx}(5x^3 - 3x^2 + 2x + 10)$$

$$= 5\times 3x^{3-1} - 3\times 2x^{2-1} + 2\times 1 + 0$$

$$= 15x^2 - 6x + 2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(15x^2 - 6x + 2)$$

$$= 15\times 2x^{2-1} - 6\times 1 + 0$$

$$= 30x - 6$$

6.8 DERIVATIVES OF VARIOUS STANDARD FUNCTIONS

1. $\frac{d}{dx}(k) = 0$, k is any constant
2. $\frac{d}{dx}(x) = 1$
3. $\frac{d}{dx}(x^n) = nx^{n-1}$
4. $\frac{d}{dx}(e^x) = e^x$
5. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$
6. $\frac{d}{dx}(a^x) = a^x \log_e a$

6.9 EXAMPLES OF ONE VARIABLE DERIVATIVE

Find $\frac{dy}{dx}$ of the following functions:

1) $y = 5x^4$

$$\frac{dy}{dx} = 4 \times 5.x^{4-1} = 20x^3$$

2) $y = -6x^5$

$$\frac{dy}{dx} = -6 \times 5.x^{5-1} = -30x^4$$

3) $y = 5x^{-2}$

$$\frac{dy}{dx} = 5 \times (-2)x^{-2-1} = -10x^{-3} = -\frac{10}{x^3}$$

4) $y = 5e^{3x}$

$$\frac{dy}{dx} = 5 \times 3.e^{3x} = 15e^{3x}$$

5) $y = \frac{1}{x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-2}) = -2.x^{-2-1} = -2^{x-3} = -\frac{2}{x^3}$$

6) $y = \frac{9}{10}x^{-5}$

$$\frac{dy}{dx} = \frac{9}{10}(-5)x^{-5-1} = \frac{9}{2}x^6$$

7) $y = (\sqrt[3]{x})^4$

$$\therefore y = x^{\frac{4}{3}}$$

$$\frac{dy}{dx} = \frac{4}{3}x^{\frac{4}{3}-1} = \frac{3}{4}x^{\frac{1}{3}}$$

8) $y = x^2(x-3)$

$$\therefore y = x^3 - 3x^2$$

$$\frac{dy}{dx} = (x^3 - 3x^2) = \frac{d}{dx}(x^3) - \frac{d}{dx}(3x^2)$$

$$= 3x^{3-1} - 3x^{2-1} = 3x^2 - 3x$$

CHECK YOUR PROGRESS

Find the first order derivative of the following.

(a) $\dfrac{x^3 + x^2 - x + 2}{x}$ (b) $\dfrac{2x+3}{3x+4}$

(c) $\dfrac{x}{a^2 + x^2}$

6.10 LET US SUM UP

- In this unit, we have discussed the concept of differentiation. We found several rules to find the derivatives of various functions.
- We have also discussed the application of differentiation, primarily to one variable case.

6.11 FURTHER READING

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5) Kandoi, Balwant (2011). *Mathematics for Business and Economics with Application*; New Delhi: Himalaya Publishing House.

6.12 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: $\frac{d}{dx}\left(\frac{x^3+x^2+x+2}{x}\right) \quad = \frac{d}{dx}\left(\frac{x^3}{x}+\frac{x^2}{x}+\frac{x}{x}+\frac{2}{x}\right)$

$$= \frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(x)-\frac{d}{dx}(1)+2\frac{d}{dx}\left(x^{-1}\right)$$

$$= 2x + 1 - 0 + 2(-1)x^{-2}$$

$$= 2x + 1 - \frac{2}{x^2}$$

Ans to Q No 2: $\frac{d}{dx}\left(\frac{2x+3}{3x+4}\right) = \frac{(3x+4)\frac{d}{dx}(2x+3)-(2x+3)\frac{d}{dx}(3x+4)}{(3x+4)^2}$

$$= \frac{(3x+4)(2+0)-(2x+3)(3+0)}{(3x+4)^2}$$

$$= \frac{(6x+8)-(6x+9)}{(3x+4)^2}$$

$$= \frac{-1}{(3x+4)^2}$$

Ans to Q No 3: $\frac{d}{dx}\left(\frac{x}{a^2+x^2}\right) = \frac{\left(a^2+x^2\right)\frac{d}{dx}(x)-x\frac{d}{dx}\left(a^2+x^2\right)}{\left(a^2+x^2\right)^2}$

$$= \frac{\left(a^2+x^2\right)1-x(2x)}{\left(a^2+x^2\right)^2}$$

$$= \frac{a^2+x^2-2x^2}{\left(a^2+x^2\right)^2}$$

$$= \frac{a^2-x^2}{\left(a^2+x^2\right)^2}$$

6.13 MODEL QUESTIONS

Q 1: Determine dy/dx for the following functions:

(a) $y = (x^2+5)(2x^3 - 4x+ 3)$ (b) $y = 3x^2 (3x + 5x^2)$

$(c) y = x^2 e^x$ $(d) y = \dfrac{x^2+1}{x-1}$ $(e) y = \dfrac{e^x}{1+x}$

$(f) y = (x+2)(x+1)^2$ $(g) y = \dfrac{3x}{x-5}$

$(h) y = 16x^2 + 15x$ $(i) y = \dfrac{1 \pm \sqrt{x}}{1-\sqrt{x}}$

Q 2: Determine the second order derivative for the following functions:

$(a) y = (3x-2)(x^2+10x)$ $(b) y = \dfrac{5x^2-2}{x^2-5x}$

$(c) y = x^3 - 6x^2 + 9x$ $(d) y = 10x^5 - 4x^3 + 5x - 2$

*** ***** ***

UNIT 7: PARTIAL DERIVATIVES AND TOTAL DERIVATIVES

UNIT STRUCTURE

7.1 LEARNING OBJECTIVES

After going through this unit, you will be able to :

- acquire knowlege about the partial derivative, total derivative and total differentials
- solve the problems of partial and total differentiation.

7.2 INTRODUCTION

In the previous unit, we have discussed the concept of derivative and also has come across derivative of one variable cases. In this unit, we shall dicuss the concepts of partial and total differentiation. At the end of the unit, you will be able to solve the problems of partial and total differentiation. In the next unit, we shall utilise these concepts to solving a few problems in Economics.

7.3 FUNCTION OF SEVERAL VARIABLES

We are familiar with function of a single variable. But in nature there arise situations where we have to deal with functions of more than one variable.

Suppose we want to forecast the weather this weekend in your town. We construct a formula for the temperature as a function of several environmental variables, each of which is not predictable. Now, we would like to see how our weather forecast would change as one particular environmental factor changes, holding all the other factor constant.To do this investigation, we would use the concept of partial derivatives.

For example, the area of a rectangle A=xy which is a function of the variable x(length) and variable y (breadth). Also, the volume of cuboid V=xyz which is a function of the variables x(length), y(breadth) and z(height).

If three variables u, y, z are so related that the value of u depends upon the values of x and y,then u is called a function of two variables x and y and this is denoted by $u = f(x, y)$. Here u is called the dependent variable while x and y are called independent variables.

Hence, $u = f(x_1, x_2)$ is a function of two variables where u is dependent variable and x_1, x_2 are indepedent variables.

Also, $u = f(x_1, x_2, x_3)$ is a function of three variables where u is dependent variable and x_1, x_2, x_3 are independent variables and $u = f(x_1, x_2, x_3, x_4)$ is a function of four variables where u is dependent variable and x_1, x_2, x_3, x_4 are indepedent variables.

In general, a function of the form $u = f(x_1, x_2, x_3, \cdots, x_n)$ is a function of n variables where u is dependent variable and $x_1, x_2, x_3, \cdots, x_n$ are independent variables.

Examples :

(i) $u = f(x, y) = x^2 + 2xy + y^2$ is a function of two variables x, y.

(ii) $u = f(x, y, z) = x^2 + 3xy + yz^2 + z^3$ is a function of three variables x,y,z.

7.4 PARTIAL DERIVATIVES

If a derivative of a function of several independent variables is found with respect to any of them, keeping the others as constants, it is said to be a **Partial Derivative**. The process of finding the partial derivatives of a function of more than one independent variables is called **Partial differentiation.**

Let $z = f(x, y)$ be a function of two variables.

The partial derivative of z with respect to x , denoted by

$\frac{\partial z}{\partial x}$ or $\frac{f(x,y)}{\partial x}$ or f_x ,is defined by

$\frac{\partial z}{\partial x} = \lim\limits_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$, provided the limit exists.

Also,the partial derivative of z with respect to y, denoted by

$\frac{\partial z}{\partial y}$ or $\frac{\partial f(x,y)}{\partial y}$ or f_y is defined by

$\frac{\partial z}{\partial y} = \lim\limits_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$,provided the limit exists.

$\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called **first order partial derivatives of z.**

Example 7.1: Given Let $f(x,y) = xy^2$. Find the partial derivatives using the limit theorems.

Solution: Given, $f(x,y) = xy^2$

Then , $f_x(x,y) = \lim\limits_{h \to 0} \frac{(x+h)y^2 - xy^2}{h}$, $f_y(x,y) = \lim\limits_{h \to 0} \frac{x(y+h)^2 - xy^2}{h}$

$$= \lim_{h \to 0} \frac{hy^2}{h} \qquad\qquad = \lim_{h \to 0} \frac{2xyh + xh^2}{h}$$

$$= y^2 \qquad\qquad = \lim_{h \to 0}(2xy + xh)$$

$$= 2xy$$

Example 7.2**:** Given $U = 5x - 6y + 8$. Find partial derivatives of fx and fy.

Solution: Given, $U = 5x - 6y + 8$.

$$\therefore f_x = \frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(5x - 6y + 8)$$

$$= 5 - 0 + 0 = 5.$$

$$\textit{Similarly}, f_y = \frac{\partial U}{\partial y} = \frac{\partial}{\partial y}(5x - 6y + 8)$$

$$= 0 - 6 + 0 = -6.$$

Example 7.3**:** Find partial derivatives of $U = 4x^2 + 4xy + y^2$.

Solution: Given, $U = 4x^2 + 4xy + y^2$.

$$\therefore f_x = \frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(4x^2 + 4xy + y^2)$$

$$= 4 \times 2x^{2-1} + 4 \times 1 \times y + 0 = 8x + 4y$$

$$Similarly, f_y = \frac{\partial U}{\partial y} = \frac{\partial}{\partial y}(4x^2 + 4xy + y^2)$$

$$= 0 + 4 \times x \times 1 + 2 \times y^{2-1} = 4x + 2y$$

Example 7.4: Find the partial derivatives of $y = x_1^{\ 3} + 2x_1 x_2^{\ 2} + 3x_2^{\ 2}$

Solution: Given, $y = x_1^{\ 3} + 2x_1 x_2^{\ 2} + 3x_2^{\ 2}$

$$\therefore fx_1 = \frac{\partial y}{\partial x_1} = \frac{\partial}{\partial x_1}(x_1^{\ 3} + 2x_1 x_2^{\ 2} + 3x_2^{\ 2})$$

$$= 3x_1^{\ 3-1} + 2x_2^{\ 2} \times 1 + 0 = 3x_1^{\ 2} + 2x_2^{\ 2}$$

$$Similarly, fx_2 = \frac{\partial y}{\partial x_2} = \frac{\partial}{\partial x_2}(x_1^{\ 3} + 2x_1 x_2^{\ 2} + 3x_2^{\ 2})$$

$$= 0 + 2x_1 \times 2x_2^{\ 2-1} + 3 \times 2x_2^{\ 2-1} = 4x_1 x_2 + 6x_2$$

Example 7.5: Find the partial derivatives of $Z = \dfrac{x+y}{2x+5y}$

Solution: Given, $Z = \dfrac{x+y}{2x+5y}$

$$\therefore f_x = \frac{(2x+5y)\frac{\partial}{\partial x}(x+y) - (x+y)\frac{\partial}{\partial x}(2x+5y)}{(2x+5y)^2}$$

$$= \frac{(2x+5y) - (x+y) \times 2}{(2x+5y)^2} = \frac{2x+5y-2x-2y}{(2x+5y)^2} = \frac{3y}{(2x+5y)^2}$$

$$Similarly, f_y = \frac{(2x+5y)\frac{\partial}{\partial y}(x+y) - (x+y)\frac{\partial}{\partial y}(2x+5y)}{(2x+5y)^2}$$

$$= \frac{(2x+5y) - (x+y) \times 5}{(2x+5y)^2} = \frac{2x+5y-5x-5y}{(2x+5y)^2} = -\frac{3x}{(2x+5y)^2}$$

Functions of more than two variables : Partial derivatives can also be defined for functions of three or more variables.

Thus if $u = f(x, y, z)$, then the partial differential coefficient of u w.r.t x

i.e. $\dfrac{\partial u}{\partial x}$ is obtained by differentiating u w.r.t x keeping y and z as constants.

i.e. $\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}=f_x(x,y,z)=\lim_{h\to 0}\frac{f(x+h,y,z)-f(x,y,z)}{h}$, provided the limit exists.

Similarly,the partial differential coefficient of u w.r.t y i.e. $\frac{\partial u}{\partial y}$ is obtained by differentiating u w.r.t y keeping x and z as constants

i.e., $\frac{\partial u}{\partial y}=\frac{\partial f}{\partial y}=f_y(x,y,z)=\lim_{h\to 0}\frac{f(x,y+h,z)-f(x,y,z)}{h}$,provided the limit exists.

Again, the partial differential coefficient of u w.r.t zi.e. $\frac{\partial u}{\partial z}$ is obtained by differentiating u w.r.t z keeping x and y as constants.

i.e., $\frac{\partial u}{\partial z}=\frac{\partial f}{\partial z}=f_z(x,y,z)=\lim_{h\to 0}\frac{f(x,y,z+h)-f(x,y,z)}{h}$

In general, if u is a function of n variables, $u=f(x_1,x_2,.......,x_n)$, its partial derivative w.r.t the variable x_i is given by

$$\frac{\partial u}{\partial x_i}=f_{x_i}(x_1,x_2,.......,x_n)$$

$$=\lim_{h\to 0}\frac{f(x_1,x_2,....,x_i+h,x_{i+1},....,x_n)-f(x_1,x_2,.......,x_n)}{h}$$, provided the limit exists.

Second order partial differential coefficients

If $z=f(x,y)$, then $\frac{\partial z}{\partial x}$ or f_x and $\frac{\partial z}{\partial y}$ or f_y are themselves functions of x and y and can be again differentiated partially.

We define $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$, $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$ and $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$ as the second order partial derivatives of u and these are as denoted by

$\frac{\partial^2 u}{\partial x\partial y}$, $\frac{\partial^2 u}{\partial y\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ respectively.

Higher order partial derivatives : Partial derivatives of higher order of a function are calculated by successive differentiation.

We denote the third order partial derivatives by

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right)=\frac{\partial^3 f}{\partial x^3}=f_{xxx},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right)=\frac{\partial^3 f}{\partial y\partial x^2}=f_{yxx},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x\partial y}\right)=\frac{\partial^3 f}{\partial x^2\partial y}=f_{xxy}\text{ ,etc.}$$

and, in general, the n-th partial derivatives are

$$\frac{\partial}{\partial x}\left(\frac{\partial^{n-1} f}{\partial x^{n-1}}\right)=\frac{\partial^n f}{\partial x^n}=f_{x^n}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^{n-1} f}{\partial x^{n-1}}\right)=\frac{\partial^n f}{\partial y\partial x^{n-1}}=f_{yx^{n-1}}\text{ etc.}$$

Illustrative Examples :

Example 7.6 **:** If $z=x^2y+xy^3$, find out the first order partial derivatives.

Solution : We have $z=x^2y+xy^3$

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left(x^2y+xy^3\right)$$

$$=\frac{\partial}{\partial x}\left(x^2y\right)+\frac{\partial}{\partial x}\left(xy^3\right)$$

$$=2xy+y^3$$

Also, $$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left(x^2y+xy^3\right)$$

$$=\frac{\partial}{\partial y}\left(x^2y\right)+\frac{\partial}{\partial y}\left(xy^3\right)$$

$$=x^2+3xy^2$$

Example 7.7 **:** If $z=\log\left(x^2+y^2\right)^{1/2}$, then find out $\frac{\partial z}{\partial x}$ and $\frac{\partial x}{\partial y}$.

Solution : Given $z=\log\left(x^2+y^2\right)^{1/2}$

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left(\log\left(x^2+y^2\right)^{1/2}\right)$$

$$=\frac{1}{2\left(x^2+y^2\right)}.2x$$

$$=\frac{x}{x^2+y^2}$$

Again, $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\log\left(x^2 + y^2\right)^{1/2}\right)$

$$= \frac{1}{2\left(x^2 + y^2\right)}.2y$$

$$= \frac{y}{x^2 + y^2}.$$

Example 7.8: If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.

Solution : Holding y constant and differentiating w.r.t x, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so $$f_x(2,1) = 3.2^2 + 2.2.1^3 = 16$$

Similarly, holding x constant and differentiating w.r.t y, we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2,1) = 3.2^2.1^2 - 4.1 = 8$$

Example 7.9 : If $u = \log\left(x^2 + y^2 + z^2\right)$, show that $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} + z.\frac{\partial u}{\partial z} = 2$

Solution : We have $u = \log\left(x^2 + y^2 + z^2\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2 + z^2} \times (2x)$$

$$x.\frac{\partial u}{\partial x} = x.\frac{2x}{x^2 + y^2 + z^2} = \frac{2x^2}{x^2 + y^2 + z^2} \quad \text{... (1)}$$

Also, $$\frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2 + z^2} \times (2y)$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2 + z^2}$$

$$y.\frac{\partial u}{\partial y} = \frac{2y^2}{x^2 + y^2 + z^2} \quad \text{...(2)}$$

and $$\frac{\partial u}{\partial z} = \frac{1}{x^2 + y^2 + z^2} \times (2z)$$

$$\frac{\partial u}{\partial z} = \frac{2z}{x^2 + y^2 + z^2}$$

$$z.\frac{\partial u}{\partial z} = \frac{2z^2}{x^2 + y^2 + z^2} \quad \text{... (3)}$$

Adding (1), (2) and (3), we get

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} + z.\frac{\partial u}{\partial z} = \frac{2\left(x^2 + y^2 + z^2\right)}{x^2 + y^2 + z^2} = 2.$$

7.5 TOTAL DIFFERENTIATION

Total Derivatives : If $u = f(x, y)$ where $x = \phi_1(t)$ and $y = \phi_2(t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}.$$

$\frac{du}{dt}$ is called the total differential coefficient of u w.r.t 't'

while $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are partial derivatives of u w.r.t x and y respectively.

The technique of partial derivative gives the rate of change in $U = \{U = f(x, y)\}$ when small changes are made in x {or in y} while y {or x} remains unchanged. But if both x and y change simultaneously, the problem is dealt with the technique of total differentials.

Proof : Let there be a small increment δt in t and let the corresponding increments in u, x and y be δu, δx and δy respectively. Then we have

$u = f(x, y)$ ... (1)

and $u + \delta u = f(x + \delta x, y + \delta y)$ ... (2)

$$\therefore \quad \delta u = f(x + \delta x, y + \delta y) - f(x, y)$$

$$= \left[f(x + \delta x, y + \delta y) - f(x, y + \delta y)\right] + \left[f(x, y + \delta y) - f(x, y)\right]$$

Dividing by δt, we get

$$\frac{\delta u}{\delta t} = \left\{\frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta t}\right\} + \left\{\frac{f(x, y + \delta y) - f(x, y)}{\delta t}\right\} \quad \text{... (3)}$$

Let $\delta t \to 0$ so that $\delta x \to 0$ and $\delta y \to 0$.

Now, from (3), we get

$$\lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \lim_{\partial t \to 0} \frac{\partial x}{\partial t}$$

$$+ \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \lim_{\partial t \to 0} \frac{\delta y}{\delta t} \quad \text{... (4)}$$

since δx and δy tend to zero with δt, therefore (4) becomes

$$\frac{du}{dt}=\frac{\partial u}{\partial x}.\frac{dx}{dt}+\frac{\partial u}{\partial y}.\frac{dy}{dt}.$$

Note: (1) If $u=f(x,y,z)$, where $x=\phi_1(t), y=\phi_2(t), z=\phi_3(t)$, then

$$\frac{du}{dt}=\frac{\partial u}{\partial x}.\frac{dx}{dt}+\frac{\partial u}{\partial y}.\frac{dy}{dt}+\frac{\partial u}{\partial z}.\frac{\partial z}{\partial t}$$

(2) The above result can be extended to any finite number of variables.

If $u=f(x_1,x_2,\cdots,x_n)$ where $x_1=\phi_1(t), x_2=\phi_2(t),\ldots,x_n=\phi_n(t)$. then

$$\frac{du}{dt}=\frac{\partial u}{\partial x_1}.\frac{dx_1}{dt}+\frac{\partial u}{\partial x_2}.\frac{dx_2}{dt}+\cdots+\frac{\partial u}{\partial x_n}.\frac{dx_n}{dt}$$

(3) If u is a function of x and y, i.e., $u=f(x,y)$ where y is a function of x, then $\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}.\frac{dy}{dx}$.

Deductions :(1) If $u=f(x,y)$ where $x=f_1(t_1,t_2)$ and $y=f_2(t_1,t_2)$, then we have

$$dZ=\frac{(x^2+y^2)d(x^2-y^2)-(x^2-y^2)d(x^2+y^2)}{(x^2+y^2)^2}$$

$$=\frac{(x^2+y^2)(2xdx-2ydy)-(x^2-y^2)(2xdx+2ydy)}{(x^2+y^2)^2}$$

$$=\frac{2x^3dx+2xy^{22}dx-2x^2ydy-2y^3dy-2x^3dx-2x^2ydy+2xy^2dx+2y^3dy}{(x^2+y^2)^2}$$

$$=\frac{-4x^2ydy+4xy^2dx}{(x^2+y^2)^2}$$

and $$\frac{\partial u}{\partial t_2}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_2}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_2}$$

ILLUSTRATIVE EXAMPLES :

Example 7.10: Find total differentials of $Z = 3x^2 + xy - 2y^3$.

Solution : Given, $Z = 3x^2 + xy - 2y^3$.

$$
\begin{aligned}
dZ &= d(3x^2) + d(xy) - d(2y^3) \\
&= 3 \times 2xdx + xdy + ydx - 2 \times 3y^{3-1}dy \\
&= 6xdx + xdy + ydx - 6y^2dy
\end{aligned}
$$

Example 7.11: Find total differentials of $y = 10 + 2x + 3x^2$

Solution : Given, $y = 10 + 2x + 3x^2$

$$
\begin{aligned}
dy &= d(10) + d(2x) + d(3x^2) \\
&= 0 + 2dx + 6xdx \\
&= 2dx + 6xdx
\end{aligned}
$$

Example 7.12: Find total differentials of $Z = (x^2 + y)(2x - y^2)$

Solution : Given, $Z = (x^2 + y)(2x - y^2)$

$$
\begin{aligned}
dZ &= (x^2 + y)d(2x - y^2) + (2x - y^2)d(x^2 + y) \\
&= (x^2 + y)(2dx - 2ydy) + (2x - y^2)(2xdx + dy) \\
&= 2x^2dx + 2ydx - 2x^2ydy - 2y^2dy + 4x^2dx - 2xy^2dx + 2xdy - y^2dy) \\
&= (6x^2 + 2y - 2xy^2)dx + (-2x^2 - 3y^2 + 2x)dy
\end{aligned}
$$

Example 7.13: Find total differentials of $y = \dfrac{2x^2}{10 + x}$

Solution : Given, $y = \dfrac{2x^2}{10 + x}$

$$
\begin{aligned}
dy &= \frac{(10 + x) \times d(2x^2) - 2x^2 \times d(10 + x)}{(10 + x)^2} \\
&= \frac{(10 + x)(2 \times 2 \times x) - 2x^2(1)}{(10 + x)^2} \\
&= \frac{40x + 4x^2 - 2x^2}{(10 + x)^2} \\
&= \frac{40x + 2x^2}{(10 + x)^2}
\end{aligned}
$$

Example 7.14: Find total differentials of $Z = \dfrac{x^2 - y^2}{x^2 + y^2}$

Solution : Given, $Z = \dfrac{x^2 - y^2}{x^2 + y^2}$

$$dZ = \frac{(x^2 + y^2)d(x^2 - y^2) - (x^2 - y^2)d(x^2 + y^2)}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2)(2xdx - 2ydy) - (x^2 - y^2)(2xdx + 2ydy)}{(x^2 + y^2)^2}$$

$$= \frac{2x^3dx + 2xy^2dx - 2x^2ydy - 2y^3dy - 2x^3dx - 2x^2ydy + 2xy^2dx + 2y^3dy}{(x^2 + y^2)^2}$$

$$= \frac{-4x^2ydy + 4xy^2dx}{(x^2 + y^2)^2}$$

Example 7.15: If $u = f(x - y, y - z, z - x)$, prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

Solution : Let $x - y = X$, $y - z = Y$, $z - x = Z$

Then $u = f(X, Y, Z)$ where X, Y, Z are functions of x, y, z.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}.\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial x}$$

$$= \frac{\partial u}{\partial X}.(1) + \frac{\partial u}{\partial Y}.(0) + \frac{\partial u}{\partial Z}.(-1)$$

$$= \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} \quad \dots (1)$$

Similarly, $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial X}.\dfrac{\partial X}{\partial y} + \dfrac{\partial u}{\partial Y}\dfrac{\partial Y}{\partial y} + \dfrac{\partial u}{\partial Z}.\dfrac{\partial Z}{\partial y}$

$$= \frac{\partial u}{\partial X}.(-1) + \frac{\partial u}{\partial Y}(1) + \frac{\partial u}{\partial Z}.(0)$$

$$= -\frac{\partial u}{\partial X}. + \frac{\partial u}{\partial Y} \quad \dots (2)$$

and $\dfrac{\partial u}{\partial z} = -\dfrac{\partial u}{\partial Y} + \dfrac{\partial u}{\partial Z} \quad \dots (3)$

Adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

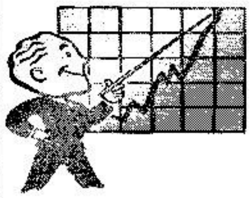
Example 7.16: Find $\frac{dz}{dt}$ when $z = xy^2 + x^2 y$, $x = at^2$, $y = 2at$.

Solution : $\frac{\partial z}{\partial x} = y^2 + 2xy$, $\frac{\partial z}{\partial y} = 2xy + x^2$

$$\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$$

Now, we have $\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$

$$= (y^2 + 2xy)2at + (2xy + x^2)2a$$
$$= (4a^2t^2 + 4a^2t^3)2at + (4a^2t^3 + a^2t^4)2a$$
$$= a^3(16t^3 + 10t^4)$$



CHECK YOUR PROGRESS

Q 1: If $u = x^5 y^4$ where $x = t^2$ and $y = t^3$, find $\frac{du}{dt}$.

7.6 LET US SUM UP

- In this unit, we have discussed the concept of partial differentiation.
- We have also learn to solve the problems of partial differentiation.

7.7 FURTHER READING

1) Baruah, S. *Basic Mathematics and Its Applications in Economics*, Trinity Press Pvt Ltd.

2) Chiang, A.C. & Wainright, K. *Fundamental Methods of Economic Analysis*, McGrawHill Education (India)

3) Madnani, B C, Mehata & G. M. *Mathematics for Economists*, Sultan Chand & Sons

7.8 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No. 1 : Here u is a function of x and y where x and y are the functions of t.

Again, u is also a function of single variable t.

$$\therefore \quad \frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

$$= \left(5x^4 y^4\right).2t + \left(4x^5 y^3\right).3t^2$$

$$= 10.x^4 y^4 t + 12x^5 y^3 t^2$$

$$= 10.t^8 t^{12} t + 12t^{10} t^9 t^2$$

$$= 22t^{21}$$

7.9 MODEL QUESTIONS

Q 1: Find the total differentiation of $u = x^2 + 2y^2$.

Q 2: Find the total differentiation of $z = x^2 + y^2 + 3xy$.

Q 3: Find the total differentiation of $z = (x^2 + y)\,(2x\text{-}y^2)$

Q 4: Determine dy/dx for the following functions:

(a) $y = (x^2+5)\,(2x^3 - 4x + 3)$

(b) $y = 3x^2\,(3x + 5x^2)$

Q 5: For the production function $Q = 40F + 3F^2 + F^{3/3}$, calculate marginal and average production functions.

Q 6: Find marginal productivity for the total productivity $Y = (x^2+2)(x+3)$.

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UNIT 8 : ECONOMIC APPLICATIONS OF DERIVATIVES

UNIT STRUCTURE

8.1 LEARNING OBJECTIVES

After going through this unit, you will be able to -

- acquire knowledge of the different functions in Economics where differential calculus is used
- apply differential calculus to economic problems.

8.2 INTRODUCTION

In the earlier two units, we discussed the concept of differential calculus. In this unit, we shall look into a few areas of its applications in

Economics. In the field of Economics, differential calculus is frequently used to find marginals from totals; e.g, in finding marginal case-or revenu from total cast or revenue. The concept is also frequently used in finding elasticities, among others.

8.3 ECONOMIC APPLICATIONS OF DERIVATIVES

In this section, we shall look into a few economic applications of derivatives.

8.3.1 Elasticity of Demand (Ed)

Elasticity of demand can be expressed as;

$$\text{Ed} = \frac{\text{Proportionate change in quantity demand}}{\text{Proportinate change in price}}$$

$$= \frac{\frac{dQ}{Q}}{\frac{dP}{P}}$$

$$\text{Ed} = \frac{dQ}{dP}.\frac{P}{Q}$$ ($\therefore$ P and Q are – vely related)

or $|\text{Ed}| = \left|\frac{dQ}{dP}.P/Q\right|$

We can express ed in following way also.

$$\text{Ed} = \frac{d(\log Q)}{d(\log P)}$$

or $\text{Ed} = \dfrac{AR}{AR - MR}$

If Ed>1, then the commodity has elastic demand

Ed<1, inelastic demand

Ed=1, then the commodity has Unitary elasticity of demand.

8.3.2 Deriving the Relationship between AR, MR and ed

We know;

$$TR = AR \times Q \text{ ———— (i)}$$

Deriving (i) w.r. to 'Q' we get,

$$\frac{dTR}{dQ} = AR\frac{dQ}{dQ} + Q\frac{dAR}{dQ}$$

$$\Rightarrow MR = AR + Q.\frac{dAR}{dQ}$$

$$\therefore MR = P + Q\frac{dP}{dQ}(\therefore AR = P = \text{Price})$$

$$\Rightarrow MR = P\left(1 + \frac{Q}{P}\frac{dP}{dQ}\right)$$

$$\Rightarrow MR = P\left(1 + \frac{1}{P/Q\frac{dQ}{dP}}\right)$$

$$\Rightarrow MR = P\left(1 - \frac{1}{ed}\right)\left(\because Ed = -\frac{dQ}{dP}P/Q\right)$$

$$\Rightarrow MR = P - P/Ed$$

$\Rightarrow AR/Ed = AR - MR$ ($\therefore$ AR=P as mentioned earlier)

$$\Rightarrow \frac{AR}{AR - MR} = Ed$$

$$\therefore Ed\frac{AR}{AR - MR}$$

8.3.3 Deriving the Relationship between AC and MC

We know;

TC = AC x Q ———————— (i)

Deriving (i) w.r. to 'Q' we get;

$$\frac{d}{dQ}TC = \frac{d}{dQ}(AC \times Q)$$

$$\Rightarrow MC = AC\frac{dQ}{dQ} + Q\frac{dAC}{dQ}$$

$$\Rightarrow MC = AC + Q.\text{Slope of AC}$$

$$\Rightarrow Q.\text{Slope of AC} = MC - AC$$

$$\therefore \text{Slope of } AC = \frac{1}{Q}(MC - AC) \quad \text{———} \quad \text{(ii)}$$

From (ii) we derive the relationship between AC and MC as follows:

(i) When $MC>AC$ then slope of $AC>0$

(ii) When $MC<AC$ then slope of $AC = 0$

(iii) When $MC < AC$ then slope of $AC<0$

8.3.4 Utility Function

A utility (U) function for two commodities can be expressed as:

$U = f(x,y)$ where u = utility

x and y are two goods

Marginal utility :

$MU_x = \frac{\partial u}{\partial x}$, here y is constant

where, MU_x = marginal utility of x

$MU_y = \frac{\partial u}{\partial y}$, here x is constant

where, MU_y = Marginal utility of y

8.3.5 Indifference Curve (IC) Function

An IC function can be expressed as:

$U = f(x,y) = C$ where c is constant.

Slope of IC = $-\frac{dy}{dx} = \frac{MU_x}{MU_y}$

Note : IC possesses two characteristics which are :

(i) IC is downward sloping is $\frac{dy}{dx} < 0$.

(ii) IC is convex to the origin is $\frac{d^2y}{dx^2} > 0$

8.3.6 Production function :

A Production function for two factors of poduction can be expressed as:

Q=f (L, K) where; Q = Output, L and K are inputs Labour and capital respectively.

Marginal product :

$MP_L = \frac{\partial Q}{\partial L}$, here K is constant

where; MP_L = Marginal Product of labour

$MP_K = \frac{dQ}{\partial K}$, here L is constant.

where; MP_k = Marginal product of capital.

8.3.7 Iso-Quant (IQ) Function

An Iso-Quant (IQ) function can be expressed as:

Q = f(L,K) = C where C is constant.

Slope of $IQ = -\frac{dK}{dL} = \frac{MP_L}{MP_K}$

Note : IC possesses two characteristics which are :

(i) IQ is downward sloping is $\frac{dK}{dL} < 0$.

(ii) IQ is convex to the origin is $\frac{d^2K}{dL^2} > 0$.

8.3.8 Relationship between MR, MP and MPR

Let us consider the two- input production function as :

Q = f (L, K) where; L = Labour input

K = Capital input.

Here, $MP_L = \frac{\partial Q}{\partial L}$, here K is constant.

where = Marginal physical product of labour.

$MP_K = \frac{\partial Q}{\partial K}$, here L is constant

where MP_K = Marginal physical product of capital.

Note : Marginal product (MP) and Marginal Physical Product (MPP) are similar.

Again; Total Revenue (TR) is a function of quantity (Q) which can be written as;

TR = f (Q)

Here, $MR = \frac{dTR}{dQ}$

Now; $MRP_L = MP_L \times MR$

and $MRP_K = MP_K \times MR$

8.3.9 Consumption Function

Consumption (C) is a function of disposable income (yd), which can be written as;

c = f (yd) where Yd = Y – T

Y = Income

T = Tax

Marginal Propensity to consume (MPC) :

$$MPC = \frac{dc}{dy}$$

Note : 0<MPC<1

8.3.10 Saving Function

Saving (s) is a function of income (y) which can be written as;

s = f (y)

Marginal Propensity to Save (MPS) :

$$MPS = \frac{ds}{dy}$$

It is important to note that MPC+MPS=1.

8.3.11 Import Function

Import (M) is a function of income (y) which can be written as;

M = f (y)

Margainal Proensity to Import (MPM)

$$MPM = \frac{dM}{dy}$$

8.3.12 Elasticity of Supply (E_s)

Elasticity of supply can be expressed as:

$$E_s = \frac{\text{Proportionate change in quantity Supply}}{\text{Proportionate change in Price}}$$

$$= \frac{dQs}{Qs} / \frac{dp}{p}$$ where; Qs = Quantity Supply

P = price of the commodity

$$\therefore E_s = \frac{dQs}{dp} \cdot \frac{P}{Qs}$$

We can express e_s in the following way also;

$$\therefore E_s = \frac{d(\log Qs)}{d(\log P)}$$

8.4 NUMERICAL PROBLEMS

In this section, we shall discuss a few numerical examples.

Example 8.1 : The demand function of a firm is given as $Q = 30 – 5P – P^2$.

Find the elasticity of demand and MR when P = 2

Solution: Given, demand function $Q = 30 - 5P - P^2$.

Now when P = 2, we get

$$Q = 30 - 5(2) - (2)^2$$

$$= 30 - 10 - 4$$

$$= 16.$$

Now,

$$Q = 30 - 5P - P^2$$

$$\therefore \frac{dQ}{dP} = -5 - 2P$$

Putting P = 2, we get,

$$\frac{dQ}{dP} = -5 - 2(2)$$

$$= -9.$$

(i) $$E_d = -\frac{dQ}{dP} \times \frac{P}{Q}$$

$$= -(-9) \times \frac{P}{Q}$$

$$= 9 \times \frac{2}{16}$$

$$= \frac{18}{16} = 1.125.$$

(ii) We know, $MR = AR\frac{(Ed-1)}{(Ed)}$

$$MR = P\frac{(1.125-1)}{(1.125)}$$

$$= 2\left(\frac{125}{1125}\right)$$

$$= 2\ (0.11)$$

$$= 0.22.$$

Example 8.2: : The deamand fucntion of a firm is given as $Q = 100 - 4P - 2P^2$. Calculate price elasticity of demand when values of P are 2, 5 and 10 respectively and interpret.

Solution: Given demand function $Q = 100 - 4P - 2P^2$ ——— (i)

$$\therefore \frac{dQ}{dP} = \frac{d}{dP}(100 - 4P - 2P^2)$$

$$= -4 - 4P \text{ ——— (ii)}$$

We know that $Ed = (-)\left(\frac{P}{Q}\right) \times \left(\frac{dQ}{dP}\right)$

(i) Now putting P = 2 in (i) we get,

$$Q = 100 - 4(2) - 2(2)^2$$

$$= 100 - 8 - 8$$

$$= 84$$

$\therefore \frac{dQ}{dP} = -4 - 4(2)$ [From equation (ii)]

$$= -4 - 8$$

$$= -12.$$

$$\therefore Ed = (-)\left(\frac{P}{Q}\right) \times \left(\frac{dQ}{dP}\right)$$

$$= (-)\frac{2}{84}(-12)$$

$$= \frac{24}{84} = \frac{2}{7} = 0.286$$

As |Ed| = 0.29 > 1. So, we can say that at P = 2, the demand is relatively inelastic. Thus, at P = 2, corresponding to a unit charge is price, the quantity demanded (Q) changes by 0.29 or by 29%.

(ii) Putting P = 5 in equation (i), we get

$$Q = 100 - 4(5) - 2(5)^2$$
$$= 100 - 20 - 50$$
$$= 30.$$

And, $\frac{dQ}{dP} = -4 - 4(5)$ [From equation (ii)]

$$= -4 - 20$$
$$= -24.$$

$$\therefore Ed = (-)\left(\frac{P}{Q}\right)\left(\frac{dQ}{dP}\right)$$

$$= (-)\left(\frac{5}{30}\right)(-24)$$

$$= \frac{5 \times 24}{30}$$

$$= 4.$$

As, Ed = 4>1, we can say that at P = 5, the demand is relatively elastic. Thus at P = 5, corresponding to a unit change in price, the quantity demanded of Q changes 4 times a by 400%.

(iii) Putting P = 10 in equation (i), we get

$$Q = 100 - 4(10) - 2(10)^2$$
$$= 100 - 40 - 200$$
$$= -140.$$

$\therefore \frac{dQ}{dP} = (-4) - 4(10)$ [From equation (ii)]

$$= -4 - 40$$
$$= -44.$$

$$\therefore Ed = (-)\left(\frac{P}{Q}\right)\left(\frac{dQ}{dP}\right)$$

$$= (-)\left(\frac{10}{-140}\right)(-44)$$

$$= -\frac{10 \times 44}{140}$$

$$= -\frac{22}{7} = -3.14$$

Or |Ed| = 3.14

As |Ed| = 3.14>1, we can say that at P = 10, the demand is relatively elastic.

Thus, at P = 10, corresponding to a unit change in price, quantity demanded 3.14 times or 314 percent.

Example 8.3 : Supply function of a firm is given as P = 4 + 5Q^2. Find elasticity of supply at P = 9, P = 6, P = 4 and P = 3 and interpret.

Solution: Given the supply function

$$P = 4 + 5Q^2 \text{ ——————— (i)}$$

$$\Rightarrow 5Q^2 = P - 4$$

$$\Rightarrow Q^2 = \frac{1}{5}(P - 4) - \text{(ii)}$$

$$\Rightarrow Q = \pm\sqrt{\frac{1}{5}(P - 4)}$$

[– sign is ignored as it does not make any sense in Economics.]

Now, $$\frac{dP}{dQ} = \frac{d}{dQ}(4 + 5Q^2)$$

$$= 10Q \text{ ———— (iii)}$$

and $$\frac{dQ}{dP} = \frac{1}{10Q}$$

We know, $$Es = \frac{P}{Q}\frac{dQ}{dP}$$

$$= \left(\frac{4 + 5Q^2}{Q}\right)\left(\frac{1}{10Q}\right)$$

$$= \frac{1}{10}\left(\frac{4 + 5Q^2}{Q^2}\right) \text{ ———— (iv)}$$

(i) When P = Q

$$Q^2 = \frac{1}{5}(9 - 4) \text{ ———— [From equation (ii)]}$$

$$= \frac{1}{5} \times 5$$

$$= 1.$$

$\therefore Q = 1$. (– sign is ignored).

$$\therefore Es = \frac{1}{10}\left\{\frac{4+5(1)^2}{(1)^2}\right\}$$ [From (iv)]

$$= \frac{1}{10} \times 9 = 0.9. < 1.$$

As Es is 0.9<1, we can say that at P = 9, elasticity of supply is relatively in elastic. Thus, at P = 9, corresponding to a unit change in price, the quantity supplied changed by 0.9 or 90%.

(ii) When P = 6,

$$Q^2 = \frac{1}{5}(6-4)$$

$$= \frac{2}{5}$$

$$\therefore Q = \pm\sqrt{\frac{2}{5}}$$

$$= \pm\sqrt{0.40}$$

$$= \pm 0.63 = 0.63$$ [Negative sing is ignored].

$$\therefore Es = \frac{1}{10}\left\{\frac{4+5\left(\frac{2}{5}\right)}{\frac{2}{5}}\right\}$$ [From equation (iv]

$$\frac{1}{10}\left(6 \times \frac{5}{2}\right)$$

$$= \frac{15}{10} = 1.5 > 1.$$

As Es is 1.5>1, we can say that at P = 6, elasticity of supply is relatively elastic. Thus at P = 6, corresponding to a unit change in price, the quantity supplied changes by 1.5 times, or by 150%.

(iii) When P = 4

$$Q^2 = \frac{1}{5}(4-4)$$

= 0

Q = 0

$$\therefore \frac{dP}{dQ} = 10(0) = 0$$

$$\frac{dQ}{dP} = \frac{1}{0} \text{ or } \infty$$

and $$\therefore Es = \frac{1}{0}\left[\frac{4+5(10)^2}{(0)^2}\right]$$

Thus, at P = 4, Es is perfectly (or infinitely) elastic)

(iv) At P = 3,

$$Q^2 = \frac{1}{5}(3-4)$$

$$= \frac{-1}{5}$$

Or, $$Q = \sqrt{-\frac{1}{5}}$$

$Q = \sqrt{-\frac{1}{5}}$ is a non-real quantity.

$\therefore$ At p = 3, Es is not defined in real terms.

Example 8.4 : The total utility function of two commodities x and y givemn as $u = 2x^3y + 3xy^2 + 3x + 3y$. Find marginal utility of y at x = 2 and y = 3.

Solution: Given, $u = 2x^3y + 3xy^2 + 3x + 3y$

$\therefore$ Marginal utility of $y = \frac{\partial u}{dy}$

$$= \frac{\partial}{dy}(2x^3y + 3xy^2 + 3x + 3y)$$

$$= 2x^3 + 6xy + 3$$

For x = 2 and y = 3

Marginal utility of $y = 2(2)^3 + 6(2)(3) + 3$

$= 16 + 36 + 3$

$= 55.$

Example 8.5: The total utility function of two goods x and y is given as $U = 5x^2y + 2xy^3 + 3x + 9y$. Find marginal utilities of x and y at x = –3 and y=2.

Solution: Given total utility function $U = 5x^2y + 2xy^3 + 3x + 9y$.

$\therefore$ Marginal utility of x $(MU_x) = \frac{\partial u}{\partial x}$

$$= \frac{\partial}{\partial x}\left(5x^2y + 2xy^3 + 3x + 9y\right)$$

$$= 10xy + 2y^3 + 3 \qquad \text{——————— (i)}$$

Marginal utility of y $(MU_y) = \dfrac{\partial u}{\partial y}$

$$= \frac{\partial}{dy}(5x^2 + 2xy^3 + 3x + 9y)$$

$$= 5x^2 + 6xy^2 + 9 \qquad \text{——————— (ii)}$$

Putting x = 3 and y = 2 in (i), we get

$$MU_x = 10(3)(2) + 2(2)^3 + 3$$

= 60 + 16 + 3

= 79

Similarly, putting x = 3 and y = 2 in (ii), we get

$$MU_y = 5(3)^2 + 6(3)(2)^2 + 9$$

= 45 + 72 + 9

= 126.

Example 8.6: The production function of a firm is given as $Q = K^2 + 2KL + L^3$, where Q stands for Qunatity, L for labour and K for capital. Compute the marginal product of capital at K = 3 and L = 2

Solution: Given the production function $Q = K^2 + 2KL + L^3$.

Now, marginal product of capital $(MP_K) = \dfrac{\partial Q}{\partial K}$

$$= \frac{\partial}{\partial x}\left(K^2 + 2KL + L^3\right)$$

= 2K + 2L.

∴ Marginal product of capital at K = 3 and L = 2

= 2 (3) + 2 (L)

= 6 + 4

= 10.

Example 8.7: The production function of a firm is given as $Q = 40F + 3F^2 - F^3/3$. Find out marginal and average production functions.

Solution: Given, the production function $Q = 40F + 3F^2 - F^3/3$

(i) Marginal production function $= \dfrac{d}{dF}(Q)$

$$= \frac{d}{dF}\left(40F + 3F^2 - F^3/3\right)$$

$$= 40 + 6F - \frac{1}{3}3F^2$$

$$= 40 + 6F - F^2$$

(ii) Average production function $= \frac{Q}{F}$

$$= \frac{40F + 3F^2 - F^3/3}{F}$$

$$= 40 + 3F - F^2/3$$

Example 8.8: The consumption function is given by $C = 1000 - \frac{5000}{3+y}$.

Find the following :

(i) Marginal propensity to consume when y = 97.

(ii) Marginal propensity to save when y = 97.

Solution: Given consumption function $C = 1000 - \frac{5000}{3+y}$.

$$= 1000 - 5000\ (3+y)^{-1}$$

(i) $MPC = \frac{dc}{dy}$

$$= \frac{d}{dy}\left\{1000 - 5000(3+y)^{-1}\right\}$$

$$= 0 - (-1)5000(3+y)^{-}$$

$$= \frac{5000}{(3+y)^2}$$

Now when y = 97, $MPC = \frac{5000}{(3+97)^2}$

$$MPC = \frac{5000}{(3+97)^2}$$

$$= \frac{5000}{10,000}$$

$$= \frac{1}{2}$$

$$= 0.5$$

(ii) The saving function is defined as S = Y – C

or $\quad S = Y - 1000 + 5000\,(3+y)^{-1}$

$$\therefore \text{MPS} = \frac{ds}{dy}$$

$$= \frac{d}{dy}\left\{y - 1000 + 5000(3+y)^{-1}\right\}$$

$$= 1 - 0 + (-1)5000(3+y)^2$$

$$= 1 - \frac{5000}{(3+y)^2}$$

Now when y = 97, $\quad \text{MPS} = 1 - \frac{5000}{(3+97)^2}$

$$= 1 - \frac{5000}{10000}$$

$$= 1 - 0.5$$

$$= 0.5$$

CHECK YOUR PROGRESS

Q 1: Given demand function Q = 200 – 4P. Find price elasticity of demand.

Q 2 : Given MR = 15 and elasticity of demand is 4. Find AR.

8.5 LET US SUM UP

- In this unit, we have discussed some of the applications of differenttial calculus in Economics.

8.6 FURTHER READING

1) Agarwal, D.K. (2012). *Business Mathematics*, New Delhi: Vrindra Publication (p) Ltd.
2) Baruah, S. (2011). *Basic Mathematics and Its Application in Economics*, New Delhi: Trinity Press Pvt. Ltd.
3) Bose, D. (2004). *Mathematical Economics*; New Delhi: Himalaya Publishing House.
4) Chiang, A.C. & (2010). *Fundamental Methods of Economics Analysis*, MC Graw Hill Education India.
5) Kandoi, Balwant (2011). *Mathematics for Business and Economics with Application*; New Delhi: Himalaya Publishing House.

8.7 ANSWER TO CHECK YOUR PROGRESS

Ans to Q.No.1: Try yourself.

Ans to Q.No.2: We know

$$AR = MR\left[\frac{Ed}{Ed-1}\right]$$

$$= 15\left[\frac{4}{4-1}\right]$$

$$= 15 \times \frac{4}{3}$$

$$= 20.$$

8.8 MODEL QUESTIONS

Q 1: Find the elasticity of demand and AR, at P = 3, if the demand function $q = 32 - 4p - p^2$.

Q 2: Find the elasticity of demand and MR for the demand function $P = -2q^2 + 18$ for q = 1.

Q 3: Given the demand curve P = (10 – 2D) $(20 - D)^2$. Find the elasticity of demand for D = 1 and D = 4.

In jthis case Find AR, if MR is Rs 18 and the elasticity is 9.

Q 4: Find the Marginal Utilities for the following functions :

i) $U = 9x^3 + 2x^2 + x - 7$

ii) $U = 3x^3 - 3x + 4$

iii) $U = x^2 + y^2 + 2xy^2$

iv) $U = 2x^2 + 3x^2 - 2x^2y^2$

v) $U = 2x^2y + 3xy - 3x$.

Q 5: The total cost function is : $C = 2Q - 2Q^2 + Q^3$

(C – total cost and Q – total quanity produced)

Find (i) Average cost function.

(ii) Martginal cost function.

Also verify that at the minimum of average cost, average cost is equal to marginal cost.

Q 6: Given the following demand curves, demonstrate the relationship between marginal revenue and elasticity of demand

(i) p = 1.47 – 0.1x (ii) p = 86 – 25x

Q 7: The total cost functions are :

(i) $C = 2Q - 2Q^2 + Q^3$

(ii) $C = 4Q - Q^2 + 2Q^3$

Find the (i) *AC* function

(ii) *MC* function

(iii) At what level of output *AC* is minimum?

(iv) Verify that at minimum average cost *AC* = *MC*.

Q 8: The total cost of producing a product is C = 3x + 200 and 50 units are produced. Find (i) Fixed Cost (ii) Variable Cost (iii) Total Cost (iv) Variable Cost per unit (v) Average Cost per unit.

Q 9: The Average Cost function for a commodity is given by $AC = \frac{2}{3}x + 15.2$. Find the Total and Marginal Cost.

Q 10: Given the Revenue Function R = 40x – $2x^2$ – 20, find the Average Revenue and Marginal Revenue.

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UNIT 9 : MATRICES

UNIT STRUCTURE

9.1 LEARNING OBJECTIVES

After going through this unit, you will be able to :

- understand the definition of a Matrix
- describe different types of Matrix
- learn matrix operations e.g. addition, subtraction and multiplication
- learn about adjoint, inverse and rank of Matrices

9.2 INTRODUCTION

Matrix is one of the most powerful tools of modern mathematics and is also widely used in Economics. Matrix were introduced by the English Mathematician Arthur Cayley in 1858.Matrices are initially connected with linear transformation. Matrix theory now occupies important position in Physics, Economics, Statistics, Engineering etc.

9.3 CONCEPT OF A MATRIX

A set of mn numbers (real or complex) arranged in the form of a rectangular array having m rows and n columns is called an mxn matrix (to be read as 'm' by 'n' matrix).

An m x n matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

The compact representation of the above matrix is

$A = [a_{ij}]$, i = 1, 2, ... m, j = 1, 2, ... n.

or, simply, $A = [a_{ij}]_{mxn}$. Sometimes, the braket [ ] can be replaced by ( ) .

The general element of the matrix A is a_{ij} which belongs to the i-th row and j-th column. a_{ij} is sometimes denoted by (i,j)th element of the matrix. Here i, the first suffix denotes the number of row and j, the second suffix denotes the number of column in which the element a_{ij} occurs.

Note :

1. A matrix having m-rows and n-columns is called a matrix of order mn (read as 'm' by 'n') or simply m x n matrix.

2. In a matrix, the number of rows need not be equal to the number of columns.

3. A matrix is denoted by the capital letters A, B, C… etc. whereas any elements of a matrix is denoted by small letters such as

$a_{ij}, b_{ij}, c_{ij}, \cdots\cdots$ etc.

Illustrative example :

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 5 \end{bmatrix}_{2\times3} \text{ or } \begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & 5 \end{pmatrix}_{2\times3}$$

is a 2 x3 matrix i.e. it has 2 rows and 3 columns. For A,

a_{12} = (1,2)th element = 2

a_{23} = (2,3)th element = 5

a_{13} = (1,3)th element = 0

$$B = \begin{bmatrix} 1+i & 2 & \frac{1}{3} \\ 4 & 0 & 1 \\ \sqrt{3} & 2-i & \sqrt{2} \end{bmatrix}_{3\times3}$$

is a 3x3 matrix. For B,

b_{11} = 1+ i, $b_{31} = \sqrt{3}$, b_{22} =0 etc.

9.4 TYPES OF MATRIX

There are different types of matrices. We will discuss one by one.

(i) Row Matrix : Any 1x n matrix which has only one row and n column is called a row matrix.

For example,

A = [2, 5,- 3, 0, $\sqrt{3}$] is a row matrix order 1×5.

In general, B = $[a_{ij}]_{1\times n}$ is a row matrix of order 1x n

(ii)**Column Matrix :** Any m x1 matrix which has m rows and only one column is called a column matrix.

For example,

$$B = \begin{bmatrix} 6 \\ 0 \\ -3 \\ \sqrt{2} \end{bmatrix}$$ is a column matrix of order 4×1

In general, $B = [b_{ij}]_{m\times 1}$ is a column matrix of order $m\times 1$.

(iii) **Square Matrix :** A matrix whose numbers of rows is equal to the number of columns is called a square matrix.

For the matrix A= $[a_{ij}]_{m\times n}$, if m = n, then the matrix A is said to be square matrix of order m.

Example :

(a)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$ is a square matrix of order 2.

(b)

$$B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 5 & 0 & 1 \end{bmatrix}$$ is a square matrix of order 3

In a square matrix A = $[a_{ij}]_{m\times m}$, the elements a_{11}, a_{22}, a_{33} ...... are called the **Diagonal Elements** and the line along which they lie is called the **Principal Diagonal** of the matrix.

(iv)**Diagonal Matrix :** A square matrix A is said to be a diagonal matrix if all its non-diagonal elements be zero.

Examples:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

are diagonal matrices of order 2 and 3 respectively.

(v)**Scalar Matrix :** A diagonal matrix (i.e. all non-diagonal elements being zero) where all the diagonal elements are equal is called a Scalar Matrix.

Examples :

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

are two scalar matrices of order 2 and 3 respectively.

Thus, the square matrix $A = [a_{ij}]_{m \times m}$ is a scalar matrix if

$a_{ij} = 0$, when $i \neq j$

$a_{ij} = k$(say) when $i = j$

(vi) **Unit (or Identity) Matrix :** A square matrix in which the diagonal elements are unity and non-diagonal elements are all zero is called a unit (or Identity)matrix.

Examples :

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Unit matrices are denoted by I.

I_2 and I_3 are Unit matrices of order 2 and 3 respectively.

Thus, the square matrix $A = [a_{ij}]_{m \times m}$ is a unit matrix if

$a_{ij} = 0$ when $i \neq j$

$a_{ij} = 1$ when $i = j$

(vii) **Null Matrix or Zero Matrix :** A matrix of order m x n in which all the elements are zero is called a null matrix (a zero matrix). It is denoted by O.

Thus,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ are all zero matrices of order 2 x2 and 4x3

respectively.

(viii) **Upper Triangular Matrix and Lower Triangular Matrix:** Let A be a square matrix of order n.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \ldots a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots a_{2n} \\ a_{31} & a_{32} & a_{33} \ldots a_{3n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1} & a_{n2} & a_{n3} \ldots a_{nn} \end{bmatrix}$$

The diagonal elements of A are a_{11}, a_{12} ,….., a_{nn} i.e. a_{ij} for i= j.

All these elements of A above the principal diagonal are a_{12}

a_{13}……. a_{1n}, a_{23}……. a_{2n}............ etc. i.e. a_{ij} for $i < j$

All these elements of A below the principal diagonal are a_{21}, a_{31},

a_{32}, …….a_{n1}, a_{n2}, ……. i.e. a_{ij} for $i > j$.

If all a_{ij} for $i > j$ be zeros in a square matrix A, then A is called **Upper Triangular matrix.**

Example :

$$A = \begin{bmatrix} 3 & 1 & 2 & 6 \\ 0 & 2 & -4 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$ is Upper Triangular Matrix

If all a_{ij} for $i < j$ be zeros is a square matrix A, the A is called **Lower Triangular Matrix.**

Example :

$$A=\begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & -1 & 4 & 0 \\ -2 & 1 & 3 & 5 \end{bmatrix} \text{ is Lower Triangular Matrix}$$

Remarks : A diagonal matrix is both upper and lower triangular.

9.5 EQUALITY OF MATRICES

Two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ are said to be equal if

(i) they are of the same size or order

(ii) each element of one matrix is equal to the corresponding element of the other matrix i.e. $a_{ij} = b_{ij}$ for all i and j.

If A and B are equal, then we write A = B.If A and B are not equal, then we write $A \neq B$.

Example :

1. If $A=\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}_{2\times2}, B=\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}_{2\times2}$ then A = B .

2. If $A=\begin{bmatrix} 1^2 & 3^2 \\ 2^2 & 4^2 \end{bmatrix}_{2\times2}, B=\begin{bmatrix} 1 & 9 \\ 4 & 16 \end{bmatrix}_{2\times2}$ then A = B.

3. If $A=\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}_{2\times2}, B=\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}_{2\times3}$ then $A \neq B$.

4. If $\begin{bmatrix} x & a \\ y & b \\ z & c \end{bmatrix}_{3\times2} = \begin{bmatrix} -3 & 0 \\ 1 & 2 \\ 4 & \sqrt{3} \end{bmatrix}_{3\times2}$

then, x = - 3, y = 1, z = 4, a = 0, b = 2, c = $\sqrt{3}$

Remarks: The inequality of two matrices arised due to the following reasons :

(i) This orders may not be equal.

(ii) Elements in the corresponding places may not be equal.

CHECK YOUR PROGRESS

Q 1: If a matrix has 10 elements what are the possible orders it can have?

Q 2: Find the values of a, b, c, d when

$$\begin{pmatrix} 2a+b & 5c-d \\ a-2b & 4c+d \end{pmatrix} = \begin{pmatrix} 4 & 11 \\ -3 & 25 \end{pmatrix}$$

Q 3: Are the following matrices equal where

$$A = \begin{pmatrix} 2 & 4 & 3 \\ 1 & 0 & 5 \\ 0 & 0 & 6 \end{pmatrix},\ B = \begin{pmatrix} 2 & 4 & 3 \\ 1 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

9.6 ADDITION AND SUBTRACTION OF MATRICES

Let A = $[a_{ij}]_{mxn}$ and B = $[b_{ij}]_{mxn}$ be two matrices of same order mxn. Then their sum(or difference) denoted by A+B (or A-B) is defined as another matrix C= $[c_{ij}]$ of the order m xn such that any element of C is the sum (or difference) of the corresponding element of A and B.

i.e. $c_{ij} = a_{ij}+b_{ij}$ for all i, j

Thus, C $=[c_{ij}] = [a_{ij}+b_{ij}]_{mxn}$

Example 9.1 : If $A = \begin{bmatrix} 5 & 1 \\ 3 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & -5 \\ -2 & 1 \end{bmatrix}$

Then, $C = A+B = \begin{bmatrix} 5+2 & 1-5 \\ 3+(-2) & 0+1 \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ 1 & 1 \end{bmatrix}$

Similarly, $D = A-B = \begin{bmatrix} 5-2 & 1-(-5) \\ 3-(-2) & 0-1 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 5 & -1 \end{bmatrix}$

Remark : Two matrices A and B are said to be conformable for addition or subtraction if they are of the same order.

Negative of a Matrix : If A be a given matrix, then the negative of A denoted by – A is the matrix whose elements are the negative of the corresponding elements of A.

Example 9.2 :

$$\text{If } A = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ -5 & 6 \end{bmatrix}, \text{then } \; -A = \begin{bmatrix} -2 & 1 \\ 0 & -4 \\ 5 & -6 \end{bmatrix}$$

Example 9.3: If $A = \begin{bmatrix} 5 & 0 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$, , find A + B and A – B

Solution :

$$A + B = \begin{bmatrix} 5 & 0 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 1 \\ 9 & 3 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 5 & 0 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5-3 & 0-1 \\ 3-6 & 1-2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -3 & -1 \end{bmatrix}$$

Example 9.4: If $A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 3 \\ 5 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 5 \\ 2 & 0 \end{bmatrix}$

Show that [A+B] + C = A + [B+C]

Solution :

$$A+B = \begin{bmatrix} 1 & 0 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 5-2 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+3 \\ 4+5 & 3-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 9 & 1 \end{bmatrix}$$

$$[A+B]+C = \begin{bmatrix} 1 & 3 \\ 9 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3 & 3+5 \\ 9+2 & 1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 \\ 11 & 1 \end{bmatrix}$$

Again, $B+C = \begin{bmatrix} 0 & 3 \\ 5 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 2 & +0 \end{bmatrix}$

$$= \begin{bmatrix} 0+3 & 3+5 \\ 5+2 & -2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 \\ 7 & -2 \end{bmatrix}$$

$$\therefore A + [B + C] = \begin{bmatrix} 1 & 0 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 8 \\ 7 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3 & 0+8 \\ 4+7 & 3-2 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 11 & 1 \end{bmatrix}$$

$$\therefore [A + B] + C = A + [B + C]$$

CHECK YOUR PROGRESS

Q 4: Is it possible to define A+B when–

(i) A has 3 rows, 2 columns and B has 2 rows, 2 columns.

(ii) A has 3 rows, 3 columns and B has 3 rows, 3 columns.

(iii) A and B are square matrices of same order.

9.7 MULTIPLICATION OF A MATRIX BY A SCALAR

Multiplication of a matrix by a scalar is defined as follows:

If $A = [a_{ij}]_{m\times n}$ is a matrix of order $m\times n$ and k is a scalar, then

$kA = k[a_{ij}]_{m\times n} = [ka_{ij}]_{m\times n}$

i.e., (i, j)th element of kA is k times the (i,j) th element of A .

Example 9.5:

If k = 3 and $A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \\ 0 & 1 \end{bmatrix}_{2\times 3}$,then

$$3A = 3\begin{bmatrix} 2 & -1 \\ -2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3\times 2 & 3\times(-1) \\ 3\times(-2) & 3\times 3 \\ 3\times 0 & 3\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -3 \\ -6 & 9 \\ 0 & 3 \end{bmatrix}_{3\times 2}$$

and,

$$\frac{1}{2}A = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ -2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\times 2 & \frac{1}{2}\times(-1) \\ \frac{1}{2}\times(-2) & \frac{1}{2}\times 3 \\ \frac{1}{2}\times 0 & \frac{1}{2}\times 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -1 & \frac{3}{2} \\ 0 & \frac{1}{2} \end{bmatrix}_{3\times 2}$$

Example 9.6:

If $A = \begin{pmatrix} 3 & 9 \\ 1 & 8 \end{pmatrix}_{2\times 2}$, $B = \begin{pmatrix} 4 & 0 \\ 7 & 2 \end{pmatrix}_{2\times 2}$

Then verify that 2(A+B) = 2A + 2B

Solution :

We have

$$A + B = \begin{bmatrix} 3 & 9 \\ 1 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 7 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3+4 & 9+0 \\ 1+7 & 8+2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 9 \\ 8 & 10 \end{bmatrix}$$

$$\therefore 2\,(A+B) = 2\begin{bmatrix}7 & 9\\ 8 & 10\end{bmatrix} = \begin{bmatrix}2\times 7 & 2\times 9\\ 2\times 8 & 2\times 10\end{bmatrix}$$

$$= \begin{bmatrix}14 & 18\\ 16 & 20\end{bmatrix}$$

Again, $2A+2B = 2\begin{bmatrix}3 & 9\\ 1 & 8\end{bmatrix} + 2\begin{bmatrix}4 & 0\\ 7 & 2\end{bmatrix}$

$$= \begin{bmatrix}2\times 3 & 2\times 9\\ 2\times 1 & 2\times 8\end{bmatrix} + \begin{bmatrix}2\times 4 & 2\times 0\\ 2\times 7 & 2\times 2\end{bmatrix}$$

$$= \begin{bmatrix}6 & 18\\ 2 & 16\end{bmatrix} + \begin{bmatrix}8 & 0\\ 14 & 4\end{bmatrix}$$

$$= \begin{bmatrix}6+8 & 18+0\\ 2+14 & 16+4\end{bmatrix}$$

$$= \begin{bmatrix}14 & 18\\ 16 & 20\end{bmatrix}$$

$\therefore 2(A+B) = 2A + 2B$ verified.

Example 9.7:

If $k = 3$, $l = -2$ and $A = \begin{bmatrix}3 & 2\\ 1 & 0\\ 2 & 1\end{bmatrix}$

Then verify that $(k+l)\,A = kA + lA$

Solution :

$$(k+l)\,A = (3-2)\begin{bmatrix}3 & 2\\ 1 & 0\\ 2 & 1\end{bmatrix} = (1)\begin{bmatrix}3 & 2\\ 1 & 0\\ 2 & 1\end{bmatrix}$$

Again,

$$kA + lA = 3A - 2A = 3\begin{bmatrix}3 & 2\\ 1 & 0\\ 2 & 1\end{bmatrix} - 2\begin{bmatrix}3 & 2\\ 1 & 0\\ 2 & 1\end{bmatrix}$$

$$= \begin{bmatrix}9 & 6\\ 3 & 0\\ 6 & 3\end{bmatrix} - \begin{bmatrix}6 & 4\\ 2 & 0\\ 4 & 2\end{bmatrix}$$

$$= \begin{bmatrix} 9-6 & 6-4 \\ 3-2 & 0-0 \\ 6-4 & 3-2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$\therefore$ (k+l) A = kA + lA.   Verified

Example 9.8: If k=4, l=2 and A = $\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$, then

Verify that k(lA) = (kl)A.

Solution :

$$k(lA) = 4(2A) = 4\begin{bmatrix} 2\times3 & 2\times2 \\ 2\times1 & 2\times0 \\ 2\times2 & 2\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 4\times2\times3 & 4\times2\times2 \\ 4\times2\times1 & 4\times2\times0 \\ 4\times2\times2 & 4\times2\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 8\times3 & 8\times2 \\ 8\times1 & 8\times0 \\ 8\times2 & 8\times1 \end{bmatrix}$$

$$= 8\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$$

= 8A

= (4×2) A

= (kl)A.   verified

Example 9.9:

If $k = 5$ and $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$

Verify that $(-k)A = -(kA) = k(-A)$

Solution :

$$(-k)\,A = (-5)\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-5)\times 3 & -(5\times 2 \\ (-5)\times 1) & -(5\times 0) \\ -(5\times 2) & -(5\times 1) \end{bmatrix}$$

$$= - \begin{bmatrix} (5)\times 3 & (5)\times 2 \\ (5)\times 1 & (5\times 0) \\ (5\times 2) & (5\times 1) \end{bmatrix} = -(kA)$$

Also, $$k(-A) = 5\begin{bmatrix} -3 & -2 \\ -1 & 0 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5\times(-3) & 5\times(-2) \\ 5\times(-1) & 5\times 0 \\ 5\times(-2) & 5\times(-1) \end{bmatrix}$$

$$= - \begin{bmatrix} 5\times 3 & 5\times 2 \\ 5\times 1 & 5\times 0 \\ 5\times 2 & 5\times 1 \end{bmatrix}$$

$$= -5\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} = -(kA) \quad \text{Verified.}$$

Example 9.10: If $A = \begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$, find 3A – 2B.

Solution :

$$3A - 2B = 3 \begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 2 \end{bmatrix} - 2 \begin{bmatrix} 4 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 9 & 6 \\ 3 & 12 & 6 \end{bmatrix} - \begin{bmatrix} 8 & 0 & -6 \\ 0 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0-8 & 9-0 & 6-(-6) \\ 3-0 & 12-2 & 6-4 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 9 & 12 \\ 3 & 10 & 2 \end{bmatrix}$$

9.8 MULTIPLICATION OF TWO MATRICES

Two matrices A and B are conformable for the product AB if and only if the number of columns of A is equal to the number of rows of B.

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$,

Then the product A,B,C is the matrix of order m×p

i.e. $AB = C = [c_{jk}]_{m\times p}$ where

$$C_{jk} = \sum_{j=1}^{n} a_{ij}\, b_{jk} = a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + \ldots\ldots\ldots\ \ldots\ldots\ldots + a_{in}\, b_{nk}$$

i.e. the (j, k) th element of the matrix C= AB is found by multiplying the corresponding element of the j-th row of A and the k-th column of B and then, adding the product.

The rule of multiplication is *row by column multiplication*.

Example 9.11:

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{2\times 2}$$

$$\text{Then } AB = \begin{bmatrix} a_{11}\, b_{11} + a_{12}\, b_{21} & a_{11}\, b_{12} + a_{12}\, b_{22} \\ a_{21}\, b_{11} + a_{22}\, b_{21} & a_{21}\, b_{12} + a_{22}\, b_{22} \\ a_{31}\, b_{11} + a_{32}\, b_{21} & a_{31}\, b_{12} + a_{32}\, b_{22} \end{bmatrix}_{3\times 2}$$

Note : 1) If the product AB exists, then it is not necessary that the product BA will also exist.

Example 9.12: A= $[2 \ \ 3]_{1\times 2}$ and B $\begin{bmatrix} 4 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix}_{2\times 3}$

= Here,

$$AB = [2 \quad 3] \begin{bmatrix} 4 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= [8+3 \quad 0+6 \quad 10+9] = [11 \ \ 6 \ \ 19]$$

But BA = $\begin{bmatrix} 4 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix} [2 \quad 3]$ which does not exist

$\therefore$ AB$\neq$BA

Matrix multiplication is not commutative

2) In the case when both A and B are square matrices of the same order then also both AB and BA are defined but still AB$\neq$BA.

Example 9.13: If A = $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and B $= \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}$

Then AB = $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2x3+1x4 & 2x5+1x1 \\ 0x3+2x4 & 0x5+2x1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 11 \\ 8 & 2 \end{bmatrix}$$

BA $= \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 3x3+5x0 & 3x1+5x2 \\ 4x2+1x0 & 4x1+1x2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 13 \\ 8 & 6 \end{bmatrix}$$

It is seen that AB $\neq$

BA **Example 9.14:**

If A = $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ and B $= \begin{bmatrix} 2 & 3 & 0 \\ -2 & 1 & 3 \end{bmatrix}$

Can you find AB and BA ?

Solution :

Here, A is of order 2×2

B is of order 2×3

$\therefore$ AB will be order 2×3

Again, B is of order 2×3

A is of order 2×2

$\therefore$BA does not exist because the number of column of B is 3 and the number of rows of A is 2.

So, they are not conformable for the product BA.

Example 9.15:

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}$, $B = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}_{2\times3}$

$C = \begin{bmatrix} 5 & 0 & 1 \\ 4 & 6 & 0 \end{bmatrix}_{2\times3}$ then show that

A(B+C) = AB + AC

Solution :

$$B+C = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 1 \\ 4 & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 1 & 3 \\ 4 & 8 & 1 \end{bmatrix}_{2\times3}$$

A(B+C)

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2} \begin{bmatrix} 8 & 1 & 3 \\ 4 & 8 & 1 \end{bmatrix}_{2\times3} = \begin{bmatrix} 1\times8+2\times4 & 1\times1+2\times8 & 1\times3+2\times1 \\ 3\times8+4\times4 & 3\times1+4\times8 & 3\times3+4\times1 \end{bmatrix}_{2\times3}$$

$$= \begin{bmatrix} 16 & 17 & 5 \\ 40 & 35 & 13 \end{bmatrix}$$

Again, $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1\times3+2\times0 & 1\times1+2\times2 & 1\times2+2\times1 \\ 3\times3+4\times0 & 3\times1+4\times2 & 3\times2+4\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 & 4 \\ 9 & 11 & 10 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 \\ 4 & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 5+2\times 4 & 1\times 0+2\times 6 & 1\times 1+2\times 0 \\ 3\times 5+4\times 4 & 3\times 0+4\times 6 & 3\times 1+4\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 12 & 1 \\ 31 & 24 & 3 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 3 & 5 & 4 \\ 9 & 11 & 10 \end{bmatrix} + \begin{bmatrix} 13 & 12 & 1 \\ 31 & 24 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 17 & 5 \\ 40 & 35 & 13 \end{bmatrix}$$

$\therefore A(B + C) = AB + AC$

Example 9.16:

$$\text{If}A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}_{2\times 2}, B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}_{3\times 2}$$

$$C = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}_{2\times 2}$$

Find A[BC] and [AB]C and show that A[BC] $\neq$ [AB]C .

Solution :

$$BC = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}_{3\times 2} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} 1\times 1+3\times 2 & 1\times 2+3\times 0 \\ 0\times 1+2\times 2 & 0\times 2+2\times 0 \\ -1\times 1+4\times 2 & -1\times 2+4\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 \\ 4 & 0 \\ 7 & 2 \end{bmatrix}$$

$$A[BC] = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ 4 & 0 \\ 7 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times7+1\times4+(-1)\times7 & 1\times2+1\times0+(-1)\times(-2) \\ 2\times7+0\times4+3\times7 & 2\times2+0\times0+3\times(-2) \\ 3\times7+(-1)\times4+2\times7 & 3\times2+(-1)\times0+2\times(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 \\ 35 & -2 \\ 31 & 2 \end{bmatrix}$$

$$\text{and } AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+1\times0+(-1)(-1) & 1\times3+1\times2+(-1)4 \\ 2\times1+0\times0+3(-1) & 2\times3+0\times2+3\times4 \\ 3\times1+(-1)\times0+2\times(-1) & 3\times3+(-1)\times2+2\times4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix}$$

$$[AB]C = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

$\therefore A[BC] \neq [AB]C$

$$= \begin{bmatrix} 2\times1+1\times2 & 2\times2+1\times0 \\ (-1)\times1+18\times2 & (-1)\times2+18\times0 \\ 1\times1+15\times2 & 1\times2+15\times0 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 35 & -2 \\ 31 & 2 \end{bmatrix}$$

$\therefore A[BC] \neq [AB]C$. Proved.

CHECK YOUR PROGRESS

Q 5: Given

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, C = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}, D = \begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

B(C+D) = BC + BD

(B+C) D = BD + CD

Q 6: For the matrices A,B,C put the numerical values of their orders, so that (A+B) C = AC + BC

Q 7: Find A and B if

$$A + B = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}, \quad A - B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Q 8: Given $A = \begin{bmatrix} 0 & 1 \\ 21 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, Show that

$A^2 = B^2 = I$ (unit matrix)

Q 9: If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$

9.9 TRANSPOSE OF A MATRIX

Let $A = [a_{ij}]_{m\times n}$. Then the n×m matrix obtained from A by changing the rows of A into columns and columns into rows is called the Transpose of matrix A and is denoted by A^{l} or A^{T}.

Therefore, A^{l} or A^{T} will have n-rows and m-columns.

Example :

If $A = \begin{bmatrix} 2 & 0 & 5 \\ 1 & 2 & 4 \end{bmatrix}_{2\times 3}$ Then,

$$A^{/} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 5 & 4 \end{bmatrix}_{3\times 2}$$

9.10 SYMMETRIC MATRIX

SYMMETRIC MATRIX :

A square matrix $A=[a_{ij}]$ is said to be symmetric if for all values of i and j, i.e, $a_{ij} = a_{ji}$

i.e. $(i,j)^{th}$ element is same as the $(j,i)^{th}$ element of A

For example

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

are symmetric matrix.

Theorem : The necessary and sufficient condition for a matrix A to be symmetric is that $A = A^{/}$

9.11 LET US SUM UP

- A rectangular array of mn numbers arranged in m-rows and n-columns and enclosed in square [ ] or a round bracket ( ), is called a matrix of order m×n (m by n).
- The numbers which constitute the matrix are called the elements of a matrix.
- The element which occurs in the i-th row and j-th column is called the (i,j) the element of the matrix and is denoted by a_{ij} .
- A matrix with exactly one row is called a row matrix. The order of a row matrix is of the type 1×n.
- A matrix with exactly one column is called a column matrix. The order of a column matrix is of the type m×1.
- A matrix A=[a_{ij}]$_{m\times n}$ is a square matrix if m=n.
- A square matrix a=[a_{ij}] is a scalar matrix if $a_{ij} = \begin{Bmatrix} k & \text{where } i = j \\ 0 & \text{wherei} \neq j \end{Bmatrix}$
- A square matrix A = [a_{ij}] is a diagonal matrix if a_{ij} = 0 where i≠j.
- A square matrix A=[a_{ij}] is a unit matrix if a_{ij} = 1 where i = j

  = 0 where i≠j ;

  A unit matrix of order n is denoted by I or I_n.
- A matrix in which all elements are zero is called a null matrix or zero matrix. A null matrix is denoted by O.
- A matrix is said to be upper triangular if a_{ij} = 0 when i>j.
- A matrix is said to be lower triangular if a_{ij} = 0 when i < j.
- Two matrix are said to be equal if and only of they are of same order and the corresponding elements are equal.
- If A and B are two matrix of same order. Then their sum or difference denoted by A + B ,A–B or B–A can be obtained.
- For matrices A and B of the same order $(A+B)' = A' + B'$
- For matrix A and B conformable to multiplication

  $$(AB)' = B'A'$$
- The matrix A is called symmetric if $A'=A$

9.12 FURTHER READING

1) Agarwal, D.K. (2012). *Business Mathematics*, New Delhi: Vrindra Publication (p) Ltd.
2) Baruah, S. (2011). *Basic Mathematics and Its Application in Economics*, New Delhi: Trinity Press Pvt. Ltd.
3) Bose, D. (2004). *Mathematical Economics;* New Delhi: Himalaya Publishing House.
4) Chiang, A.C. (2006) *Fundamental Methods of Economics Analysis*; New Delhi: MC Graw Hill Education India.
5) Kandoi, Balwant (2011). *Mathematics for Business and Economics with Application;* New Delhi: Himalaya Publishing House.

9.13 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q No 1: Find all possible orders of a matrix with 10 elements.

$10 = 1 \times 10 \qquad 10 = 5 \times 2$

$10 = 2 \times 5 \qquad 10 = 10 \times 1$

Possible orders of matrices are 1×10, 2×5, 5×2 and 10×1.

Answer to Q No 2: By equality of two matrices $\begin{pmatrix} 2a+b & 5c+d \\ a-2b & 4c+d \end{pmatrix} = \begin{pmatrix} 4 & 11 \\ 3 & 25 \end{pmatrix}$

2a+b=4   5c–d=11

a–2b=–3   4c+d=25

Solving these we get

a = 1, b = 2, c = 4, d = 9

Answer to Q No 3: For equality of two matrices, their orders and the co-rresponding elements must be same. Here, the order of the matrices are same. But (3,3)th element of $A \neq (3,3)^{th}$ element of B $\therefore$ A $\neq$ B

Answer to Q No 4:

(i) It is not possible for define A + B, since A is of order 3×2 and B is of order 2×2.

(ii) It is possible to define A+B, since A is of order 3×3 and B is of order 3×3.

(iii) It is possible to define A+B, since A and B are square matrices of the same order.

Answer to Q No 5:

$$B(C+D)=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\left(\begin{bmatrix}3 & 1\\ 4 & 2\end{bmatrix}+\begin{bmatrix}5 & 1\\ 7 & 4\end{bmatrix}\right)$$

$$=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}8 & 2\\ 11 & 6\end{bmatrix}$$

$$=\begin{bmatrix}1x8+2x11 & 1x2+2x6\\ 3x8+4x11 & 3x2+4x6\end{bmatrix}$$

$$=\begin{bmatrix}30 & 14\\ 68 & 30\end{bmatrix}$$

$$BC+BD=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}3 & 1\\ 4 & 2\end{bmatrix}+\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}5 & 1\\ 7 & 4\end{bmatrix}$$

$$=\begin{bmatrix}1x3+2x4 & 1x1+2x2\\ 3x3+4x4 & 3x1+4x2\end{bmatrix}+\begin{bmatrix}1x5+2x7 & 1x1+2x4\\ 3x5+4x7 & 3x1+4x4\end{bmatrix}$$

$$=\begin{bmatrix}11 & 5\\ 25 & 11\end{bmatrix}+\begin{bmatrix}19 & 9\\ 43 & 19\end{bmatrix}$$

$$=\begin{bmatrix}11+19 & 5+9\\ 25+43 & 11+19\end{bmatrix}=\begin{bmatrix}30 & 14\\ 68 & 30\end{bmatrix}$$

$$\therefore B(C+D)=BC+CD$$

$$\text{Again }(B+C)D=\left(\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}+\begin{bmatrix}3 & 1\\ 4 & 2\end{bmatrix}\right)\begin{bmatrix}5 & 1\\ 7 & 4\end{bmatrix}$$

$$=\begin{bmatrix}1+3 & 2+1\\ 3+4 & 4+2\end{bmatrix}\begin{bmatrix}5 & 1\\ 7 & 4\end{bmatrix}$$

$$=\begin{bmatrix}4 & 3\\ 7 & 6\end{bmatrix}\begin{bmatrix}5 & 1\\ 7 & 4\end{bmatrix}$$

$$=\begin{bmatrix}4x5+3x7 & 4x1+3x4\\ 7x5+6x7 & 7x1+6x4\end{bmatrix}$$

$$= \begin{bmatrix} 41 & 16 \\ 77 & 31 \end{bmatrix}$$

$$BD+CD= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}+\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1x5+2x7 & 1x1+2x4 \\ 3x5+4x7 & 3x1+4x4 \end{bmatrix}+\begin{bmatrix} 3x5+1x7 & 3x1+1x4 \\ 4x5+2x7 & 4x1+2x4 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 9 \\ 43 & 19 \end{bmatrix}+\begin{bmatrix} 22 & 7 \\ 34 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 19+22 & 9+7 \\ 43+34 & 19+12 \end{bmatrix}=\begin{bmatrix} 41 & 16 \\ 77 & 31 \end{bmatrix}$$

$$\therefore (B+C)D = BD+CD$$

Answer to Q No 6:

If, A is of order 4×3

Then, B is of order 4×3

and C is of order 3×2

Answer to Q No 7:

Given, $A + B = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$, $A - B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$$\therefore A+B+A-B = 2A = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}+\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 5+3 \end{bmatrix}=\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$$

$$A = \frac{1}{2}\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}=\begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

Again,

A+B – (A–B) = A+B–A+B=2B

$$= \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}-\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3 & 0-0 \\ 2-0 & 5-3 \end{bmatrix}=\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$$

$$\therefore B = \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}=\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Answer to Q No 8:

$$A^2 = AA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0x0 + 1x1 & 0x1 + 1x0 \\ 1x0 + 0x1 & 1x1 + 0x0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$B^2 = BB = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0x0 + (-i)i & 0x(-i) + (-i)x0 \\ ix0 + 0xi & ix(-i) + 0x0 \end{bmatrix}$$

$$= \begin{bmatrix} -i^2 & 0 \\ 0 & -i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Answer to Q No 9:

A^2–5A+7I=A.A–5A+7I

$$= \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - 5 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3x3 + 1(-1) & 3x1 + 1X2 \\ (-1)x3 + 2(-1) & (-1)x1 + 2x2 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8 - 15 + 7 & 5 - 5 + 0 \\ -5 + 5 + 0 & 3 - 10 + 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

9.14 MODEL QUESTIONS

Q 1: In a Matrix $\begin{bmatrix} 4 & -3 & 19 & \sqrt{2} \\ 11 & 24 & -7 & -5 \\ 1 & \sqrt{3} & 0 & 1 \end{bmatrix}$

Find (i) Order of the matrix

(ii) The number of elements

(iii) Write the elements a_{23}, a_{12}, a_{32}, a_{34}, $a_{44.}$

Q 2: If $A = \begin{bmatrix} 4 & 5 \\ 6 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 3 & -4 \end{bmatrix}$, find $A + B$ and $A - B$

Q 3: If $A = \begin{bmatrix} 3 & 2 \\ 0 & -2 \\ 1 & 0 \end{bmatrix}$, find $3A$ and $-5A$

Q 4: Find AB and BA if $A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \\ 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 5 & 3 \end{bmatrix}$

Q 5: What is the additive identity of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$?

Q 6: If $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ -1 & 3 & 5 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 \\ 1 & 3 \\ 0 & 2 \end{bmatrix}$

$C = \begin{bmatrix} 3 & 1 & 2 \\ -2 & 2 & 0 \end{bmatrix}$, find A (BC) and (AB) C

and show that A[BC] = [AB] C

Q 7: Show that A[B+C] = AB + AC

where $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 & 5 \\ -1 & 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 & -3 \\ 6 & 0 & 4 \end{bmatrix}$

Q 8: If $A = \begin{bmatrix} 4 & 3 & 7 \\ 2 & 5 & 6 \end{bmatrix}$ show that $(A')' = A$.

Q 9: If $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$, then show that $(AB)' = B'A'$

Q 10: If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then show that $A^4 = \begin{bmatrix} 1+2\times4 & -4\times4 \\ 4 & 1-2\times4 \end{bmatrix}$

Q 11: Find A^{-1} where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

Q 12: Find the inverse of the matrix $\begin{bmatrix} 4 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Q 13: If $A = \begin{bmatrix} 2 & -4 & -2 \\ 4 & 6 & 2 \\ 0 & 10 & -4 \end{bmatrix}$, find adj A

Q 14: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$, find A^{-1}

*** ***** ***

UNIT 10: DETERMINANTS

UNIT STRUCTURE

10.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define determinant
- evaluate determinants of order 2 and 3
- use the properties of determinants for evaluation of determinants
- solve determinants using Cramer's rule.

10.2 INTRODUCTION

The concept of determinants is a useful tool in solving system of linear equations in two or three variables. In this unit,we shall discusss the concept of determinants.We shall also study many properties of determinants which help in evaluation of determinants. We may use determinants to solve a system of linear equations by a method known as Cramer's rule.

10.3 DETERMINANT OF ORDER 2

Let us consider the equations $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$.

Eliminating x and y from the equations, we have $-\frac{y}{x}=\frac{a_1}{b_1}=\frac{a_2}{b_2}$

$$\Rightarrow a_1b_2-a_2b_1=0$$

which is written in compact form as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=0$

In other words, we have $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=a_1b_2-a_2b_1=0$.

The expression $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a determinant of the second order. It has two horizotal lines $a_1 \quad b_1$ and $a_2 \quad b_2$ and two vertical lines $\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix}$. The horizontal lines are called rows and the vertical lines are called columns.A determinant of second order has two rows and two columns.The numbers a_1, b_1, a_2, b_2 are called the elements of the determinant. $a_1b_2-a_2b_1$ is called the expansion or the value of the determinant.

Example 5.1: Evaluate (i) $\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix}$ (ii) $\begin{vmatrix} x+1 & x \\ x & x-1 \end{vmatrix}$ (iii) $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix}$

(iv) $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$

Solution : (i) We have $\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix}=1.(-4)-(-2).3=-4+6=2$.

(ii) $\begin{vmatrix} x+1 & x \\ x & x-1 \end{vmatrix}=(x+1).(x-1)-x.x=x^2-1-x^2=-1$.

(iii) $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix}=(x-1)(x^2+x+1)-x^3$

$$=x^3-1-x^3$$
$$=-1\cdot$$

(iv) $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}=a^2b^2-(ab)^2=0$.

Example 10.2 : Find the value of x if

(i) $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = 6$ (ii) $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = 0$

Solution : (i) We have $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = (x-3)(x+3) - x(x+1)$

$$= (x^2 - 9) - x^2 - x$$
$$= -x - 9$$

$\therefore -x - 9 = 6$

$\Rightarrow -x = 15$

$\Rightarrow x = -15$

(ii) We have $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = (2x-1)(4x+2) - (x+1)(2x+1)$

$$= 8x^2 + 4x - 4x - 2 - 2x^2 - x - 2x - 1$$
$$= 6x^2 - 3x - 3$$
$$= 3(2x^2 - x - 1)$$

$\therefore 3(2x^2 - x - 1) = 0$

$\Rightarrow 2x^2 - x - 1 = 0$

$\Rightarrow x = \dfrac{1 \pm \sqrt{1+8}}{4}$

$= \dfrac{1 \pm 3}{4}$

$= 1, -\dfrac{1}{2}$

Q 10.3. Solve for x if $\begin{vmatrix} x & 5 \\ 7 & x \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} = 0$.

Solution : Given, $\begin{vmatrix} x & 5 \\ 7 & x \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} = 0$

$\Rightarrow (x^2 - 35) + (1 - 2) = 0$

$\Rightarrow x^2 - 35 - 1 = 0$

$\Rightarrow x^2 - 36 = 0$

$\Rightarrow x^2 = 36$

$\Rightarrow x = \pm 6$

CHECK YOUR PROGRESS

Q 1: Evaluate :

(i) $\begin{vmatrix} 2 & -3 \\ 7 & 11 \end{vmatrix}$ (ii) $\begin{vmatrix} 2+\sqrt{3} & 3+\sqrt{11} \\ 3-\sqrt{11} & 2-\sqrt{3} \end{vmatrix}$

Q 2: Solve for x :

(i) $\begin{vmatrix} 2 & 3 \\ 1 & 4x \end{vmatrix} = \begin{vmatrix} 2x & -1 \\ 5 & x \end{vmatrix}$ (ii) $\begin{vmatrix} x & 3 \\ 4 & x \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 2x & 5 \end{vmatrix}$

Q 3: If $\begin{vmatrix} 3 & p \\ 4 & 6 \end{vmatrix} = 6$, find the value of p .

Q 4: Prove that $\begin{vmatrix} a+ib & c+id \\ c-id & a-ib \end{vmatrix} = a^2+b^2-c^2-d^2$

10.4 DETERMINANT OF ORDER 3

In 5.3, We have already discussed determinant of order 2. Now, we think of a determinant which has 3 rows and 3 colums.

Let us consider the equations

$a_1x+b_1y+c_1z=0$ ... (1)

$a_2x+b_2y+c_2z=0$ ... (2)

and $a_3x+b_3y+c_3z=0$ ... (3)

Soving (2) and (3), we get $\dfrac{x}{b_2c_3-b_3c_2} = \dfrac{y}{c_2a_3-c_3a_2} = \dfrac{z}{a_2b_3-a_3b_2}$

Substituting these proportional values of x, y and z in (1), we get

$$a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2)=0$$

which can be written in compact form as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

In other words, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) + b_1(a_3c_2 - a_2c_3) + c_1(a_2b_3 - a_3b_2)$$

The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called a determinant of the third order. Since it has 3 rows and 3 columns,it is called a determinant of order 3.

$a_1(b_2c_3 - b_3c_2) + b_1(a_3c_2 - a_2c_3) + c_1(a_2b_3 - a_3b_2)$ is called the expansion or the value of the determinant.

Note : A determinant of order 3 has 9 elements.

Value of a determinant :

Determinant of order three can be determined by expressing it in terms of second order determinants.This is known as expansion of a determinant along a row (or a column).There are six ways of expanding a determinant of order 3 corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3) giving the same value.

Consider the determinant of order 3

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

First we expand the given determinant along first row (R_1):

Step 1 : We multiply element a_1 of R_1 by $(-1)^{1+1}$ [$(-1)^{\text{position of the element } a_1}$] and with the second order determinant obtained by deleting the elements of first row (R_1) and first column (C_1) as a_1 lies in R_1 and C_1.

i.e., $(-1)^{1+1} a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

Step 2 : We multiply second element b_1 of R_1 by $(-1)^{1+2}$ [$(-1)^{\text{position of the element } b_1}$] and with the second order determinant obtained by deleting the elements of first row (R_1) and second column (C_2) as b_1 lies in R_1 and C_2.

i.e., $(-1)^{1+2} b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$

Step 3 : We multiply third element c_1 of R_1 by $(-1)^{1+3}$ [$(-1)^{\text{position of the element } c_1}$] and with the second order determinant obtained by deleting the elements of first row(R_1) and third column (C_3) as c_1 lies in R_1 and C_3.

i.e., $(-1)^{1+3} c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

Step 4 : The expansion of determinant is written as the sum of all the three terms obtained in step 1, 2 and 3 above and is given by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)^{1+1} a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + (-1)^{1+2} b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + (-1)^{1+3} c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 .$$

Expansion along second row (R_2):

Consider the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding along R_2, we get

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)^{2+1} a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + (-1)^{2+2} b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + (-1)^{2+3} c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= -a_2(b_1c_3 - b_3c_1) + b_2(a_2c_3 - a_3c_1) - c_2(a_1b_3 - a_3b_1)$$

$$= -a_2b_1c_3 + a_2b_3c_1 + a_1b_2c_3 - a_3b_2c_1 - a_1b_3c_2 + a_3b_1c_2 .$$

Expanding along C_1, we get

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)^{1+1} a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + (-1)^{2+1} a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + (-1)^{3+1} a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

Similarly, we can expand the determinant along R_3, C_2 and C_3.

Note : Expanding a determinant along any row or column gives same value.

Example 10.4 : Evaluate the determinant $\begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$

Solution : Expanding along R_1 we get

$$\begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix} = 1\begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} - 2\begin{vmatrix} -1 & 0 \\ 4 & 0 \end{vmatrix} + 4\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix}$$

$$= 1(0-0) - 2(0-0) + 4(-1-12)$$

$$= -52 \cdot$$

Example 10.5 : Evaluate the determinant $\Delta = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & 4 \\ 1 & -1 & -2 \end{vmatrix}$

(i) by expanding about any row

(ii) by expanding about any column.

Solution : (i) (a) Expanding about the first row

$$\Delta = 1\begin{vmatrix} 3 & 4 \\ -1 & -2 \end{vmatrix} - (-2)\begin{vmatrix} -2 & 4 \\ 1 & -2 \end{vmatrix} + 3\begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix}$$

$$= 1[3(-2) - (-1)4] + 2[(-2)(-2) - (1)(4)] + 3[(-2)(-1) - (3)(1)]$$

$$= -6 + 4 + 2(4-4) + 3(2-3)$$

$$= -2 - 3$$

$$= -5$$

(b) Expanding about the second row

$$\Delta = (-1)(-2)\begin{vmatrix} -2 & 3 \\ -1 & -2 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} - 4\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix}$$

$$= 2(4+3) + 3(-2-3) - 4(-1+2)$$

$$= 14 - 15 - 4$$

$$= -5$$

(c) Expanding about the third row

$$\Delta = 1\begin{vmatrix} -2 & 3 \\ 3 & 4 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= -8 - 9 + 4 + 6 - 2(3 - 4)$$
$$= -17 + 10 + 2$$
$$= -5$$

(ii) (a) Expanding about first column

$$\Delta = 1\begin{vmatrix} 3 & 4 \\ -1 & -2 \end{vmatrix} + 2\begin{vmatrix} -2 & 3 \\ -1 & -2 \end{vmatrix} + 1\begin{vmatrix} -2 & 3 \\ 3 & 4 \end{vmatrix}$$
$$= -6 + 4 + 2(4 + 3) + 1(-8 - 9)$$
$$= -2 + 14 - 17$$
$$= -5$$

(b) Expanding about the second column

$$\Delta = (-1)(-2)\begin{vmatrix} -2 & 4 \\ 1 & -2 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} + (-1)(-1)\begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix}$$
$$= 2(4 - 4) + 3(-2 - 3) + (4 + 6)$$
$$= -15 + 10$$
$$= -5$$

(c) Expanding about the third column

$$\Delta = 3\begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} - 4\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} + (-2)\begin{vmatrix} 1 & -2 \\ -2 & 3 \end{vmatrix}$$
$$= 3(2 - 3) - 4(-1 + 2) - 2(3 - 4)$$
$$= -3 - 4 + 2$$
$$= -5$$

10.5 PROPERTIES OF DETERMINANTS

There are some properties of determinants, which are very much useful in solving problems. Here we are going to discuss the properties only for the determinant of order 3.

Property 1 : The value of determinant remains unchanged if a rows and columns are interchanged.

i.e., $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Note : If R_i = *i*th row and C_i = *i*th column, then for interchange of row and columns, we will symbolically write $C_i \leftrightarrow R_i$.

Property 2 : If any two adjacent rows (or columns) of a determinant are interchanged, then sign of determinant changes.

i.e., $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Note : We can denote the interchange of rows by $R_i \leftrightarrow R_j$ and interchange of columns by $C_i \leftrightarrow C_j$.

Property 3 : If any two rows (or columns) of a determinant are identical (all corresponding elements are same), then value of determinant is zero.

i.e., $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Property 4 : If each element of a row (or a column) of a determinant is multiplied by a constant *k*, then its value gets multiplied by *k*.

i.e., $\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Note :The value of the determinant remains unchanged.by applying $R_i \leftrightarrow kR_i$ or $C_i \leftrightarrow kC_i$ to the determinant .

Property 5 : If to any row(or column) is added k times the corresponding elements of another row (or column), the value of the determinant remains unchanged.

i.e., $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 & c_1 + kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Note : The value of determinant remain same if we apply the operation R*i* $\leftrightarrow R_i + kR_j$ or $C_i \leftrightarrow C_i + k\,C_j$.

Property 6 : If any row (or column) is the sum of two or more elements, then the determinant can be expressed as sum of two or more determinants.

i.e., $\begin{vmatrix} a_1+d_1 & b_1+d_2 & c_1+d_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & d_2 & d_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Illustrative Examples :

Example 10.6 : Prove

that
$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Solution : $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$

(Apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$)

$$= (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$$

$$= (b-a)(c-a)(c+a-b-a)$$

$$= (b-a)(c-a)(c-b)$$

$$= (a-b)(b-c)(c-a).$$

Example 10.7 : Show that

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$$

Solution : $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

Apply $C_3 \to C_3 + C_2$

$$= \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & b+c+a \\ 1 & c & c+a+b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$$

$= (a+b+c).0$ (C_1 and C_3 are identical)

$= 0 \cdot$

Example 10.8 : Find x if $\begin{vmatrix} 1 & x & -4 \\ 5 & 3 & 0 \\ -2 & -4 & 8 \end{vmatrix} = 0$.

Solution : Expanding by 1st row

$$\begin{vmatrix} 1 & x & -4 \\ 5 & 3 & 0 \\ -2 & -4 & 8 \end{vmatrix} = 1\begin{vmatrix} 3 & 0 \\ -4 & 8 \end{vmatrix} - x\begin{vmatrix} 5 & 0 \\ -2 & 8 \end{vmatrix} + (-4)\begin{vmatrix} 5 & 3 \\ -2 & -4 \end{vmatrix}$$

$$= 1(24) - x(40) - 4(-20+6)$$

$$= 24 - 40x + 56$$

$$= -40x + 80$$

$\Rightarrow -40x + 80 = 0$

$\Rightarrow x = 2 \cdot$

Example 10.9 : Solve for x if $\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & x \\ 1 & 3 & x \end{vmatrix} = 0$.

Solution : Given , $\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & x \\ 1 & 3 & x \end{vmatrix} = 0$

$$\Rightarrow 0\begin{vmatrix} 2 & x \\ 3 & x \end{vmatrix} - 1\begin{vmatrix} x & x \\ 1 & x \end{vmatrix} + 0\begin{vmatrix} x & 2 \\ 1 & 3 \end{vmatrix} = 0$$

$\Rightarrow 0 - 1(x^2 - x) + 0 = 0$

$\Rightarrow -x^2 + x = 0$

$\Rightarrow x(1-x) = 0$

$\Rightarrow x = 0,1$.

CHECK YOUR PROGRESS

Q 5: Find the value of the following determinants:

(a) $\begin{vmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix}$ (b) $\begin{vmatrix} 4 & -1 & 5 \\ 5 & 7 & -3 \\ 3 & 4 & 1 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & -1 & 2 \\ 4 & 1 & 1 \\ 5 & -1 & 8 \end{vmatrix}$

(i) by expanding about any one row

(ii) by expanding about any one column.

Q 6: Factorise the determinants and solve the equations :

(a) $\begin{vmatrix} x & x^2 & x^2 \\ 3 & 9 & 27 \\ -1 & -2 & -3 \end{vmatrix} = 0$ (b) $\begin{vmatrix} 1 & x^3 & 2 \\ 1 & x^2 & 3 \\ 1 & x & 4 \end{vmatrix} = 0$

10.6 SOLUTION OF A SET OF LINEAR EQUATIONS BY CRAMER'S RULE

Let the system of n non-homogenous linear equations in n-unknowns linear

$$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots + a_{1n} x_n = b_1 \rightarrow (1)$$

$$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots + a_{2n} x_n = b_2 \rightarrow (2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n1} x_1 + a_{n2} x_2 + \ldots + a_{nn} x_n = b_n \rightarrow (n)$$

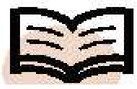
The system can be written as –

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}_{n\times 1}$$

i.e. AX = B

Determinant of the co-efficient matrix A

=|A| = D (say)



The system of writing equations within [ ] is known as Matrix. We shall discuss Matrix algebra in the next unit.

Multiplying the equations (1), (2), ..., (n) respectively by the co-factors of $a_{11}, a_{21}, \cdots$ i.e. $A_{11}, A_{21}, \cdots, A_{n1}$ and adding we get

$$A_{11}(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n) + A_{21}(a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n) + \cdots$$
$$+ A_{n1}(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n) = b_1A_{11} + b_2A_{21} + \cdots + b_nA_{n1}$$
$$\Rightarrow (a_{11}A_1 + a_{21}A_{21} + \cdots + a_{n1}A_{n1})x_1 = b_1A_{11} + b_2A_{21} + \cdots + b_nA_{n1}$$
$$\Rightarrow Dx_1 = D_1$$

Where

$$D = \begin{vmatrix} a_{11} & a_{12} \text{........} & a_{1n} \\ a_{21} & a_{22} \text{........} & a_{2n} \\ \text{..........} & \text{..........} & \text{......} \\ a_{n1} & a_{n2} \text{........} & a_{nn} \end{vmatrix} \text{ and } D_1 = \begin{vmatrix} b_1 & a_{12} \text{........} & a_{1n} \\ b_2 & a_{22} \text{........} & a_{2n} \\ \text{..........} & \text{..........} & \text{......} \\ b_n & a_{n2} \text{........} & a_{nn} \end{vmatrix}$$

D_1 is the determinant obtained from D by replacing the elements of 1st column by corresponding b's

$$\Rightarrow \frac{x_1}{D_1} = \frac{1}{D} \text{ Provided } D \neq 0$$

Similarly multiplying the equations (1), (2), ... (n) by co-factors of the elements of 2nd column of |A| and adding, we get –

$$Dx_2 = D_2 \quad \text{where} \quad D_2 = \begin{vmatrix} a_{11} & b_1 \text{.....} a_{1n} \\ a_{21} & b_2 \text{.....} a_{2n} \\ a_{n1} & b_n \text{.....} a_{nn} \end{vmatrix}$$

$$\Rightarrow \frac{x_2}{D_2} = \frac{1}{D}, D \neq 0$$

As above we will get

$$\frac{x_1}{D_1} = \frac{x_2}{D_2} = \text{..........} = \frac{x_n}{D_n} = \frac{1}{D}, D \neq 0$$

The unique solution of the given system of equation provided $D \neq 0$ in the coefficient matrix is non-singular.

i.e. the rank of the co-efficient matrix is n = number of variables.

Note : For a system of n non-homogeneous linear equations with n-unknowns

$$\frac{x_1}{D_1} = \frac{x_2}{D_2} \text{..........} \frac{x_n}{D_n} = \frac{1}{D}, D \neq 0$$

Example 10.10 : Solve the equations

$3x + y + 2z = 3$

$2x - 3y - z = -3$

$x + 2y + z = 4$

Using Cramer's rule.

Solution : We have

$$D = |A| = \begin{vmatrix} 3 & 1 & -2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$= 3(-3 + 2) - 1(2+1) + 2(4+3)$
$= -3 - 3 + 14$
$= 8 \neq 0$

$$D_1 = \begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix}$$

$= 3(-3+2) - 1(-3+4) + 2(-6+12)$
$= -3 - 1 + 12$
$= 8$

$$D_2 = \begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix}$$

$= 3(-3+4) - 3(2+1) + 2(8+3)$
$= 3 - 9 + 22 = 16$

$$D_3 = \begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix}$$

$= 3(-12+6) - 1(8+3) + 3(4+3)$
$= -18 - 11 + 21$
$= -8$

$$\therefore x = \frac{D_1}{D} = \frac{8}{8} = 1$$

$$y = \frac{D_2}{D} = \frac{16}{8} = 2$$

$$z = \frac{D_3}{D} = \frac{-8}{8} = -1$$

Example 10.11 : Solve the equations using Cramer's rule:

x+2y+3z = 6

3x–2y+z=2

4x+2y+z=7

Solution :

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix} \begin{aligned} &= 1(-2-2)-2(3-4)+3(6+8) \\ &= -4+2+42 = 40 \neq 0 \end{aligned}$$

$$D_1 = \begin{vmatrix} 6 & 2 & 3 \\ 2 & -2 & 1 \\ 7 & 2 & 1 \end{vmatrix} \begin{aligned} &= 6(-2-2)-2(2-7)+3(4+14) \\ &= -24+10+54 = 40 \end{aligned}$$

$$D_2 = \begin{vmatrix} 1 & 6 & 3 \\ 3 & 2 & 1 \\ 4 & 7 & 1 \end{vmatrix}$$

= 1(2–7)–6(3–4)+3(21–8)
= –5+6+39
= 40

$$D_3 = \begin{vmatrix} 1 & 2 & 6 \\ 3 & -2 & 2 \\ 4 & 2 & 7 \end{vmatrix}$$

= 1(–14–4)–2(21–8)+6(6+8)
= –18–26+84
= 84 – 44
= 40

$$\therefore x = \frac{D_1}{D} = \frac{40}{40} = 1$$

$$y = \frac{D_2}{D} = \frac{40}{40} = 1$$

$$z = \frac{D_3}{D} = \frac{40}{40} = 1$$

CHECK YOUR PROGRESS

Solve the following system of equations by using Cramer's rule :

Q 7: $3x + 5y = 8$, $-x+2y-z=0$, $3x-6y+4z=1$

Q 8: $x_1+x_2+x_3=7$, $x_1-x_2+x_3=2$, $2x_1-x_2+3x_3=9$

10.7 LET US SUM UP

- The value of determinant is unaltered, when its rows and columns are interchanged.
- If any two adjacent rows (columns) of a determinant are interchanged, then the value of the determinant changes only in sign.
- If the determinant has two identical rows (columns), then the value of the determinant is zero.
- If all the elements in a row or in a (column) of a determinant are multiplied by a constant $k(k > 0)$ then the value of the determinant is multiplied by k.
- The value of the determinant is unaltered when a constant multiple of the elements of any row (column), is added to the corresponding elements of a different row (column) in a determinant.
- If each element of a row (column) of a determinant is expressed as the sum of two or more terms, then the determinant is expressed as the sum of two or more determinants of the same order.

10.8 FURTHER READING

1) Agarwal, D.K. (2012). *Business Mathematics*, New Delhi: Vrindra Publication (p) Ltd.

2) Baruah, S. (2011). *Basic Mathematics and Its Application in Economics*, New Delhi: Trinity Press Pvt. Ltd.

3) Bose, D. (2004). *Mathematical Economics;* New Delhi: Himalaya Publishing House.

4) Chiang, A.C. (2006) *Fundamental Methods of Economics Analysis*; New Delhi: MC Graw Hill Education India.

5) Kandoi, Balwant (2011). *Mathematics for Business and Economics with Application;* New Delhi: Himalaya Publishing House.

10.9 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: (i) $\begin{vmatrix} 2 & -3 \\ 7 & 11 \end{vmatrix} = 22 + 21$

$= 43$.

(ii) $\begin{vmatrix} 2+\sqrt{3} & 3+\sqrt{11} \\ 3-\sqrt{11} & 2-\sqrt{3} \end{vmatrix} = \left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right) - \left(3+\sqrt{11}\right)\left(3-\sqrt{11}\right)$

$= (4-3) - (9-11)$

$= 3$.

Ans to Q No 2: (i) Given, $\begin{vmatrix} 2 & 3 \\ 1 & 4x \end{vmatrix} = \begin{vmatrix} 2x & -1 \\ 5 & x \end{vmatrix}$

$\Rightarrow 8x - 3 = 2x^2 + 5$

$\Rightarrow 2x^2 - 8x + 8 = 0$

$\Rightarrow x^2 - 4x + 4 = 0$

$\Rightarrow (x-2)^2 = 0$

$\Rightarrow x = 2,2$.

(ii) Given, $\begin{vmatrix} x & 3 \\ 4 & x \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 2x & 5 \end{vmatrix}$

$\Rightarrow x^2 - 12 = 0 + 4x$

$\Rightarrow x^2 - 4x - 12 = 0$

$\Rightarrow x^2 - 6x + 2x - 12 = 0$

$\Rightarrow x(x-6) + 2(x-6) = 0$

$\Rightarrow (x-6)(x+2) = 0$

$\Rightarrow x = -2,6$.

Ans to Q No 3: Given, $\begin{vmatrix} 3 & p \\ 4 & 6 \end{vmatrix} = 6$

$$\Rightarrow 18 - 4p = 6$$
$$\Rightarrow -4p = -12$$
$$\Rightarrow p = 3.$$

Ans to Q No 4: L.H.S $= \begin{vmatrix} a+ib & c+id \\ c-id & a-ib \end{vmatrix}$

$$= (a+ib)(a-ib) - (c+id)(c-id)$$
$$= \{a^2 - (ib)^2\} - \{c^2 - (id)^2\}$$
$$= a^2 + b^2 - c^2 - d^2$$

Ans to Q No 5: (a) (i) Expanding about the first row

$$\begin{vmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$$

$$= 2(2-1) - 1(-1-2) - 3(1+4)$$
$$= 2 + 3 - 15$$
$$= -10$$

(ii) Expanding about the first column

$$\begin{vmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & -3 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix}$$

$$= 2(2-1) - 1(-1+3) + 2(1-6)$$
$$= 2 - 2 - 10$$
$$= -10 \cdot$$

(b) Try yourself

(c) Try yourself.

Ans to Q No 6: (a) Given, $\begin{vmatrix} x & x^2 & x^2 \\ 3 & 9 & 27 \\ -1 & -2 & -3 \end{vmatrix} = 0$

$$\Rightarrow 3x\begin{vmatrix} 1 & x & x^2 \\ 1 & 3 & 9 \\ -1 & -2 & -3 \end{vmatrix} = 0$$

$$\Rightarrow 3x\begin{vmatrix} 3 & 9 \\ -2 & -3 \end{vmatrix} - 3x\begin{vmatrix} x & x^2 \\ -2 & -3 \end{vmatrix} - 3x\begin{vmatrix} x & x^2 \\ 3 & 9 \end{vmatrix} = 0$$

$$\Rightarrow 3x(-9+18) - 3x(-3x+2x^2) - 3x(9x-3x^2) = 0$$

$$\Rightarrow 3x(9+3x-2x^2-9x+3x^2) = 0$$

$$\Rightarrow 3x(x^2-6x+9) = 0$$

$$\Rightarrow 3x(x-3)^2 = 0$$

$$\Rightarrow x = 0,3$$

(b) Try yourself.

Answer Q No 7:

3x + 5y + 0z = 8
–x + 2y – z = 0
3x – 6y + 4z = 1

$$D = \begin{vmatrix} 3 & 5 & 0 \\ -1 & 2 & -1 \\ 3 & -6 & 4 \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & -1 \\ -6 & 4 \end{vmatrix} - 5\begin{vmatrix} -1 & -1 \\ 3 & 4 \end{vmatrix}$$

= 3 (8-6) – 5(–4+3)
= 6+5 = $11 \neq 0$

$$D_1 = \begin{vmatrix} 8 & 5 & 0 \\ 0 & 2 & -1 \\ 1 & -6 & 4 \end{vmatrix} = 8\begin{vmatrix} 2 & -1 \\ -6 & 4 \end{vmatrix} - 5\begin{vmatrix} 0 & -1 \\ 1 & 4 \end{vmatrix}$$

= 8(8–6) –5x1=16–5=11

$$D_2 = \begin{vmatrix} 3 & 8 & 0 \\ -1 & 0 & -1 \\ 3 & 1 & 4 \end{vmatrix}$$

$$= 3\begin{vmatrix} 0 & -1 \\ 1 & 4 \end{vmatrix} - 8\begin{vmatrix} -1 & -1 \\ 3 & 4 \end{vmatrix}$$

= 3(0+1)–8(–4+3)

= 3+8 = 11

$$D_3 = \begin{vmatrix} 3 & 5 & 8 \\ -1 & 2 & 0 \\ 3 & -6 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & 0 \\ -6 & 1 \end{vmatrix} - 5\begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix} + 8\begin{vmatrix} -1 & 2 \\ 3 & -6 \end{vmatrix}$$

= 3(2–0) –5(–1–0) +8(6–6)

= 6+5 =11

$$\therefore x = \frac{D_1}{D} = \frac{11}{11} = 1$$

$$y = \frac{D_2}{D} = \frac{11}{11} = 1$$

$$y = \frac{D_3}{D} = \frac{11}{11} = 1$$

i.e. x =1, y =1, z =1.

Answer Q No 8:

$x_1 + x_2 + x_3 = 6$

$x_1 - x_2 + x_3 = 2$

$2x_1 - x_2 + 3x_3 = 9$

$$\therefore D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = 1x\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} - 1x\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + ax\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix}$$

= (–3+1)–(3–2)+(–1+2)

= –2–1+1= $-2 \neq 0$

$$\therefore D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ 9 & -1 & 3 \end{vmatrix} = 6\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} - 1x\begin{vmatrix} 2 & 1 \\ 9 & 3 \end{vmatrix} + 1x\begin{vmatrix} 2 & -1 \\ 9 & -1 \end{vmatrix}$$

= 6(–3+1)–(6–9)+(–2+9)

= –12+3+7= – 2

$$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 2 & 9 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ 9 & 3 \end{vmatrix} - 6x \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 9 \end{vmatrix}$$

= (6–9) –6(3–2) +(9–4)

= –3 –6 +5

= –9 + 5

= – 4

$$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & -1 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 2 \\ -1 & 9 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 2 & 9 \end{vmatrix} + 6 \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix}$$

= (–9+2) –(9–4) + 6 x(–1+2)

= –7–5+6

= –12+6

= –6

$$x_1 = \frac{D_1}{D} = \frac{-2}{-2} = 1$$

$$x_2 = \frac{D_2}{D} = \frac{-4}{-2} = 2$$

$$x_3 = \frac{D_3}{D} = \frac{-6}{-2} = 3$$

10.10 MODEL QUESTIONS

Q 1: Evaluate (i) $\begin{vmatrix} 4 & 6 \\ -2 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix}$ (iii) $\begin{vmatrix} -2 & -4 \\ -1 & -6 \end{vmatrix}$

Q 2: Evaluate (i) $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \\ 1 & 2 & 4 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Q 3: Show that

$$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

Q 4: Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-a)(c-a)(a-b)(a+b+c)$$

Q 5: Show that $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$.

Q 6: Show that $\begin{vmatrix} a & b & c \\ a-c & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc$

Q 7: Prove that $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$

Q 8: Show that $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy$

Q 9: Without expanding the determinant, prove that

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Q 10: Solve using Cramer's rule

$$2x+3y+z=9$$
$$x+2y+3z=6$$
$$3x+y+2x=8$$

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